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Contents

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Index

Notations used in the talks

$\wedge^{p,q} = \bigoplus_{\begin{cases} r \leqq p \\ s \leqq q \end{cases}} I^{r,s}$
A^* = the dual of a vector space A
$A^{p,q}(X) = C^\infty(p,q)$ forms on a complex manifold X
$A^r = \bigoplus_{p+q=r}$ = polarized Hodge structure (PHS)
$(A)_{\mathbb{R}}$= real points in a complex vector space having a conjugation.
$\mathfrak{b}$ = Borel subalgebra
$\mathbb{B}$ = unit ball in $\mathbb{C}^2 \subset \mathbb{P}^2$
$\mathbb{B}^c = \mathbb{P}^2 \backslash$(closure of $\mathbb{B}$)
$\overline{\mathbb{B}}$ = unit ball with conjugate complex structure
B = Cartan-Killing form or Borel subgroup, depending on the context
$\widetilde{B}(N)$ = set of nilpotent orbits (F,N), $F \in \check{D}$
$B(N) = \widetilde{B}(N)$ modulo rescalings
$\mathbf{c}_\beta$ = Cayley transform associated to a real, non-compact root β
d_π = relative differential
$\mathcal{D}_n^\pm$ = discrete series, and their limits, for $\mathrm{SL}_2(\mathbb{R})$
D_φ = Mumford-Tate domain
$\boxtimes$ is the external tensor product
F^p = Hodge filtration bundles
$F^p_\infty = \lim_{\mathrm{Im}\, z \to \infty} \exp(zN) \cdot F$ for a nilpotent orbit (F,N)
$G = \mathbb{Q}$-algebraic group
$G_{\mathbb{R}}, G_{\mathbb{C}}$ = corresponding real and complex Lie groups
$\mathfrak{g}^\alpha, \mathfrak{h}, X_\alpha$ etc. are standard notations from Lie theory listed in Lecture 2
$\mathfrak{g}(W)$ = set of gradings associated to a filtration W
$\mathfrak{g}_N = \ker(\mathrm{ad}\, N) \cap \mathrm{Im}(\mathrm{ad}\, N)$
$\mathrm{Gr}\, B(N)$ = set of graded polarized Hodge structures associated to $B(N)$
$G_{\widetilde{\varphi}}$ = Mumford-Tate group of $(V, \widetilde{\varphi})$
G_φ = Mumford-Tate group of (V, Q, φ)
$G_{\mathcal{W}}$ = part of $G_{\mathbb{C}}$ lying over $\mathcal{W}$
$\mathrm{Gr}(n,E)$ = Grassmannian of n-planes in a complex vector space E
$\mathbb{G}(n,E)$ = Grassmannian of $\mathbb{P}^{n-1}$'s in $\mathbb{P}E$
$G_L(n,E)$ = Lagrangian Grassmannian of n-planes P in a vector space E having a bilinear form Q and with $Q(P,P) = 0$
$\mathbb{G}_L(n,E)$ = Lagrangian Grassmannian of Lagrangian $\mathbb{P}^{n-1}$'s in $\mathbb{P}E$.
$h^{p,q}$ = Hodge numbers and $f^p = \sum_{p/\geqq p} h^{p',q'}$

$H^*_{\mathrm{DR}}\big(\Gamma(M, \Omega^\bullet_\pi(F)); d_\pi\big)$ = de Rham cohomology of global, relative F-valued
holomorphic forms
$\mathcal{H}$ = upper half plane
$I^{p,q}$ = Deligne splitting of a mixed Hodge structure
$\mathcal{I}$ = incidence space
κ_μ = Kostant class
$\mathcal{L}^{p-1,q-1} = \bigoplus_{\substack{a<p \\ b<q}} I^{a,b}$
$\mathfrak{n}$ = direct sum of negative root spaces (except in the appendix to Lecture 6)
$\mathfrak{n}_c$ = direct sum of negative, compact root spaces
$\mathfrak{n}_{nc}$ = direct sum of negative, non-compact root spaces
$\mathcal{O}G_{\mathcal{W}}$ = global holomorphic functions on $G_{\mathcal{W}}$
$\mathcal{O}_{\mathbb{P}^n(k)}$ = standard line bundle over projective space
ω_Z = canonical line bundle for a complex manifold Z
Ω_μ = curvature form of $L_\mu \to D$
Ω_π = sheaf of relative differential forms
P = weight lattice
$\pi^*\mathcal{F}$ = pullback of a vector bundle
$\pi^{-1}\mathcal{F}$ = pullback of a coherent analytic sheaf
Φ^+_c, Φ^+_{nc} are positive compact, respectively non-compact roots
Φ, Φ^+ = roots, respectively positive roots
$q(\mu) = \#\{\alpha \in \Phi^+_c : (\mu, \alpha) < 0\} + \#\{\beta \in \Phi^+_{nc} : (\mu, \beta) > 0\}$
R = root lattice
$\rho = (1/2)$ (sum of positive roots)
$\mathrm{Res}_{\mathbb{C}/\mathbb{R}}$ = restriction of scalars
$s_\alpha \in W$ is reflection in the α root plane
σ_μ = Schmid class
$\mathbb{S} = \mathbb{Q}$-algebraic group given by $\{\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) : a, b \in \mathbb{Q} \text{ and } a^2 + b^2 = 1\}$
$\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ = universal enveloping algebra
$\mathcal{U}$ = cycle space $\subset \check{\mathcal{U}} = G_{\mathbb{C}}/K_{\mathbb{C}}$
V = vector spaced defined over $\mathbb{Q}$
$V_{\mathbb{R}}$ = real vectors in a complex vector space V with a conjugation σ
$V_{\mathbb{R}}, V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{R}$, $V \otimes_{\mathbb{Q}} \mathbb{C}$
$V^{p,q}$ = Hodge (p,q) spaces
$\mathbb{V}^{p,q}$ = Hodge bundles
$(V, \widetilde{\varphi})$ = general Hodge structure
W = Weyl group of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$
W_K = Weyl group of $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{t})$ = "compact" Weyl group
$(W(N), F)$ = limiting mixed Hodge structure
$\mathcal{W}$ = correspondence space included in its dual $\check{\mathcal{W}}$
χ_ζ = infinitesimal character
$Z_G(H)$ = centralizer in G of a subgroup $H \subset G$

This monograph presents topics in Hodge theory and representation theory, two of the most active and important areas in contemporary mathematics. The underlying theme is the use of complex geometry to understand the two subjects and their relationships to one another—an approach that is complementary to what is in the literature. Finite-dimensional representation theory and complex geometry enter via the concept of Hodge representations and Hodge domains. Infinite-dimensional representation theory, specifically the discrete series and their limits, enters through the realization of these representations through complex geometry as pioneered by Schmid, and in the subsequent description of automorphic cohomology. For the latter topic, of particular importance is the recent work of Carayol that potentially introduces a new perspective in arithmetic automorphic representation theory.

The present work gives a treatment of Carayol's work, and some extensions of it, set in a general complex geometric framework. Additional subjects include a description of the relationship between limiting mixed Hodge structures and the boundary orbit structure of Hodge domains, a general treatment of the correspondence spaces that are used to construct Penrose transforms and selected other topics from the recent literature.

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