Ergodic Theory
and Fractal Geometry
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Hillel Furstenberg
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Dedicated to the memory of Benoît Mandelbrot
Contents

Preface ix
Chapter 1. Introduction to Fractals 1
Chapter 2. Dimension 11
Chapter 3. Trees and Fractals 13
Chapter 4. Invariant Sets 21
Chapter 5. Probability Trees 23
Chapter 6. Galleries 27
Chapter 7. Probability Trees Revisited 31
Chapter 8. Elements of Ergodic Theory 33
Chapter 9. Galleries of Trees 35
Chapter 10. General Remarks on Markov Systems 37
Chapter 11. Markov Operator $T$ and Measure Preserving Transformation $T$ 39
Chapter 12. Probability Trees and Galleries 43
Chapter 13. Ergodic Theorem and the Proof of the Main Theorem 47
Chapter 14. An Application: The $k$-lane property 51
Chapter 15. Dimension and Energy 53
Chapter 16. Dimension Conservation 55
Chapter 17. Ergodic Theorem for Sequences of Functions 57
Chapter 18. Dimension Conservation for Homogeneous Fractals: The Main Steps in the Proof 59
Chapter 19. Verifying the Conditions of the Ergodic Theorem for Sequences of Functions 65
Bibliography 67
Index 69
Dynamics in all its variations is the study of change. In the usual physical context, change takes place within time. The objects of geometry are static and if there is any change, it is “in the eye of the beholder”. In fractal geometry this point takes on meaning, particularly in the form of changing degree of magnification and “zooming in” on an object. This suggests developing dynamical concepts appropriate to this framework.

In these notes, based on a series of lectures delivered at Kent State University in 2011, we show that ergodic theoretic concepts can be applied to the process of changing magnification to give insight to phenomena peculiar to fractals. An important step is showing how fractal dimension can be interpreted in terms of ergodic averages in an appropriate measure preserving system. The familiar phenomenon of self similarity appears as the analogue of periodicity in classical dynamics. We don’t pursue the full implications of recurrence in the geometric context, but some examples of the related Ramsey type questions are considered.

The theory developed here and the major ideas originated in the papers \([F]\) and \([F']\). It will develop that there is a close connection between dimension theory and rates of growth of trees. This is exploited in \([FW]\) where analogs of Szemerédi’s theorem are demonstrated in the context of trees.

I am indebted to Dmitry Ryabogin and Fedor Nazarov for transcribing the lectures as well as for working out many details that were not provided in the lectures as I presented them.

Hillel Furstenberg, January, 2014
Jerusalem, Israel
Bibliography


Index

A-tree, 15
k-lane property, 51
basin of attraction, 9
Birkhoff ergodic theorem, 33
boundary, 15
Cantor set, 4
Caratheodory theorem, 31
dimension conservation, 55
dimension of a tree, 15
Entropy function, 42
flat section, 16
Frostman's lemma, 13, 25
gallery, 27
gallery of trees, 35
Hausdorff dimension, 12
Hausdorff distance, 1
homogeneous, 56
Information function, 42
Julia set, 6
Mandelbrot fractal, 7
Markov operator, 37
Markov process, 32, 37
micro-set, 2
mini-set, 2
minimal, 15
minimal section, 17
Minkowski dimension, 11
Newton method, 8
probability tree, 23
Riesz theorem, 37, 39
section, 15
Sierpinski gasket, 3
star dimension, 13, 36
stationary measures, 39
stationary process, 37
stationary random process, 33
successor tree, 35
Fractal geometry represents a radical departure from classical geometry, which focuses on smooth objects that “straighten out” under magnification. Fractals, which take their name from the shape of fractured objects, can be characterized as retaining their lack of smoothness under magnification. The properties of fractals come to light under repeated magnification, which we refer to informally as “zooming in”. This zooming-in process has its parallels in dynamics, and the varying “scenery” corresponds to the evolution of dynamical variables.

The present monograph focuses on applications of one branch of dynamics—ergodic theory—to the geometry of fractals. Much attention is given to the all-important notion of fractal dimension, which is shown to be intimately related to the study of ergodic averages. It has been long known that dynamical systems serve as a rich source of fractal examples. The primary goal in this monograph is to demonstrate how the minute structure of fractals is unfolded when seen in the light of related dynamics.