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Number 132

Tensors: Asymptotic Geometry and Developments 2016–2018

J.M. Landsberg



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NSF-CBMS Regional Conference in the Mathematical Sciences on Tensors and
Their Uses in Approximation Theory, Quantum Information Theory, and
Geometry held at Auburn University, Alabama, July 24–28, 2017

Partially supported by the National Science Foundation.

The author acknowledges support from the Conference Board of Mathematical Sciences
and NSF grant number DMS 1642659. Any opinions, findings, and conclusions or
recommendations expressed in this material are those of the author and do not
necessarily reflect the views of the National Science Foundation.

2010 *Mathematics Subject Classification*. Primary 68Q17, 14L30, 15A69, 81P45.

For additional information and updates on this book, visit
www.ams.org/bookpages/cbms-132

Library of Congress Cataloging-in-Publication Data

Names: Landsberg, J.M., author. | Conference Board of the Mathematical Sciences. | National Science Foundation (U.S.)

Title: Tensors : asymptotic geometry and developments, 2016-2018/J.M. Landsberg.

Description: Providence, Rhode Island : Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, [2019] | Series: CBMS regional conference series in mathematics ; number 132 | "Support from the National Science Foundation." | Includes bibliographical references and index.

Identifiers: LCCN 2019001001 | ISBN 9781470451363 (alk. paper)

Subjects: LCSH: Calculus of tensors | Metric spaces. | Moment problems (Mathematics). | AMS: Computer science – Theory of computing – Computational difficulty of problems (lower bounds, completeness, difficulty of approximation, etc.). msc | Algebraic geometry – Algebraic groups – Group actions on varieties or schemes (quotients). msc | Linear and multilinear algebra; matrix theory – Basic linear algebra – Multilinear algebra, tensor products. msc | Quantum theory – Axiomatics, foundations, philosophy – Quantum information, communication, networks. msc.

Classification: LCC QA433.L3255 2019 | DDC 515/.63–dc23

LC record available at <https://lccn.loc.gov/2019001001>

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Preface

This book presents and expands upon material presented at a July 2017 CBMS lecture series on the geometry of tensors and applications. The expansion includes two very exciting recent developments regarding tensors:

- Y. Shitov refuted two longstanding conjectures; V. Strassen’s 1973 additivity conjecture and P. Comon’s symmetric rank conjecture. I present a detailed exposition of his refutation of Strassen’s conjecture after giving a general introduction to Shitov’s perspective that allowed resolution of these conjectures.
- Christandl-Vrana-Zuiddam introduced *quantum spectral points*, which brought together Strassen’s spectral theory (developed for the study of matrix multiplication) and quantum information theory, by employing representation theory and rational moment polytopes. This work brings together many interesting ideas and opens doors to future research. I give a detailed exposition of their theory.

In addition to these two advances, I also present the following recent developments:

- In my lectures I had included what at the time were several open questions regarding tensor network states, which I thought should be resolvable with geometric techniques. My impression turned out to be too correct—as I started writing up these notes, two of them were essentially solved [MS, MSV]. I present expositions of both, as well as an introduction to tensor network states with an emphasis on applications to solid state physics.
- An exposition of the elementary proofs in [DM18] and [EGOW17] of the limits of rank methods in proving border rank lower bounds.
- Recent approaches towards bounding the exponent of matrix multiplication that include: a description of the use of the symmetrized matrix multiplication polynomial to bound the exponent of matrix multiplication [CHI+18, Con17], the potential use symmetry groups of tensors in the laser method [CGLVb], and an exploration of Strassen’s generalization of the conjecture that $\omega = 2$ to all tight tensors [CGL+18].

Beyond the recent developments, these notes include expositions of Strassen’s laser method, Strassen’s spectral theory, basics of quantum information theory, the resolution of the quantum marginal problem by Klyatchko and independently Christandl-Mitchison, and a largely self-contained exposition of moment polytopes in projective algebraic geometry as developed by Brion.

The book is divided into three parts.

- The first part is a brief introduction to tensors, complementary to the one in [Lan12].
- The second part is a study of tensors by employing linear algebra. The topics covered are as follows:

The use of determinantal equations for bounding border rank. In the past few years, severe limits on determinantal methods were discovered. I present elementary limitations due to Derksen-Makam and Efremenko-Garg-Oliveira-Wigderson that complement the more subtle and geometric limitations discovered by Buczyński-Galzacka (following Bernardi-Ranestad) detailed in [Lan17, §10.2].

Shitov’s refutations discussed above appear in this part.

The remainder of this part is a discussion of tensor networks with an emphasis on their use in condensed matter theory. To quote [PGVWC07] Entanglement “is a blessing for *quantum information theory* - it facilitates exponential speed-ups in quantum simulation and quantum computing – it is often more a curse for *condensed matter theory* where the complexity of such systems make them hardly tractable by classical means.” For condensed matter theory and other areas discussed in this chapter that are haunted by the so-called “curse of dimensionality”, the problems are approached by reducing to linear algebra via tensor networks. This section also includes pointers to the physics literature. I warn mathematicians that the referenced papers do not always define their terms, as is considered mandatory in mathematics.

- The third part discusses the asymptotic geometry of tensors. This theory is based on Shannon’s information theory and probability. I present the requisite background material in Chapter 4. Then, in Chapter 5, I explain how ideas from information theory were used to obtain upper bounds on the exponent of matrix multiplication, and led to Strassen’s asymptotic spectra. In order to discuss the quantum spectral points in context, in Chapter 6 I give a very brief introduction to quantum information theory. In reference to the quote from [PGVWC07] above, for quantum information theory, representation theory is used to extract qualitative information about tensors in the above-mentioned “cursed” spaces. The topics of Chapters 4, 5 and 6 are brought together in Chapter 7 with the quantum spectral points. The main result regarding the quantum spectral points depends heavily on a very special case of Brion’s theorem on the rational moment polytope. I give a proof of Brion’s theorem in general, as well as a general introduction to moment maps and moment polytopes in algebraic geometry in Chapter 8.

Acknowledgements. I have gotten substantial help in this project at every step along the way. First I would like to thank C.J. Bott, Meighan Dillon, Paul Goerlach, Christian Ikenmeyer, Matt Kerr, Mateusz Michalek, Luke Oeding, Rafael Oliveira, Arpan Pal, Eric Sabo, Anna Seigal, Tim Seynnaeve, Jacob Turner, Péter Vrana, Michael Walter, and Albert Werner for numerous comments, corrections and suggestions. I am especially grateful to Luke Oeding, for suggesting, organizing,

and running the lecture series, to Luke Oeding, Jacob Turner and Péter Vrana, who read the entire text and were immensely helpful in improving it, to Yaroslav Shitov, who took a week of his time to explain his results to Michalek, Seynnaeve and myself, and to Mateusz Michalek and Tim Seynnaeve, who spent a second week with me translating Shitov's work into more geometric language.

J.M. Landsberg

Bibliography

- [Aar13] Scott Aaronson, *Quantum computing since Democritus*, Cambridge University Press, Cambridge, 2013. MR3058839
- [AB09] Sanjeev Arora and Boaz Barak, *Computational complexity*, Cambridge University Press, Cambridge, 2009. A modern approach. MR2500087
- [AFLG15] Andris Ambainis, Yuval Filmus, and François Le Gall, *Fast matrix multiplication: limitations of the Coppersmith-Winograd method (extended abstract)*, STOC'15—Proceedings of the 2015 ACM Symposium on Theory of Computing, ACM, New York, 2015, pp. 585–593. MR3388238
- [AFT11] Boris Alexeev, Michael A. Forbes, and Jacob Tsimmerman, *Tensor rank: some lower and upper bounds*, 26th Annual IEEE Conference on Computational Complexity, IEEE Computer Soc., Los Alamitos, CA, 2011, pp. 283–291. MR3025382
- [AKLT88] Ian Affleck, Tom Kennedy, Elliott H. Lieb, and Hal Tasaki, *Valence bond ground states in isotropic quantum antiferromagnets*, Comm. Math. Phys. **115** (1988), no. 3, 477–528. MR931672
- [AL70] Huzihiro Araki and Elliott H. Lieb, *Entropy inequalities*, Comm. Math. Phys. **18** (1970), 160–170. MR0266563
- [AOP09] Hirotachi Abo, Giorgio Ottaviani, and Chris Peterson, *Induction for secant varieties of Segre varieties*, Trans. Amer. Math. Soc. **361** (2009), no. 2, 767–792, DOI 10.1090/S0002-9947-08-04725-9. MR2452824
- [AV18] J. Alman and V. Vassilevska Williams, *Limits on All Known (and Some Unknown) Approaches to Matrix Multiplication*, ArXiv e-prints (2018).
- [AW18] Josh Alman and Virginia Vassilevska Williams, *Further limitations of the known approaches for matrix multiplication*, 9th Innovations in Theoretical Computer Science, LIPIcs. Leibniz Int. Proc. Inform., vol. 94, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018, pp. Art. No. 25, 15. MR3761761
- [BBPS96] Charles H. Bennett, Herbert J. Bernstein, Sandu Popescu, and Benjamin Schumacher, *Concentrating partial entanglement by local operations*, Phys. Rev. A **53** (1996), 2046–2052.
- [BCC+17] Jonah Blasiak, Thomas Church, Henry Cohn, Joshua A. Grochow, Eric Naslund, William F. Sawin, and Chris Umans, *On cap sets and the group-theoretic approach to matrix multiplication*, Discrete Anal. (2017), Paper No. 3, 27. MR3631613
- [BCHW16] F. G. S. L. Brandao, M. Christandl, A. W. Harrow, and M. Walter, *The Mathematics of Entanglement*, ArXiv e-prints (2016).
- [BCS97] Peter Bürgisser, Michael Clausen, and M. Amin Shokrollahi, *Algebraic complexity theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 315, Springer-Verlag, Berlin, 1997. With the collaboration of Thomas Lickteig. MR1440179
- [Bel64] J. S. Bell, *On the Einstein Podolsky Rosen paradox*, Phys. Phys. Fiz. **1** (1964), no. 3, 195–200, DOI 10.1103/PhysicsPhysiqueFizika.1.195. MR3790629
- [BG11] Alessandra Bernardi, Alessandro Gimigliano, and Monica Idà, *Computing symmetric rank for symmetric tensors*, J. Symbolic Comput. **46** (2011), no. 1, 34–53, DOI 10.1016/j.jsc.2010.08.001. MR2736357
- [BGO+18] Peter Bürgisser, Ankit Garg, Rafael Oliveira, Michael Walter, and Avi Wigderson, *Alternating minimization, scaling algorithms, and the null-cone problem from invariant theory*, 9th Innovations in Theoretical Computer Science, LIPIcs. Leibniz Int. Proc. Inform., vol. 94, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018, pp. Art. No. 24, 20. MR3761760

- [BILR] Grey Ballard, Christian Ikenmeyer, J.M. Landsberg, and Nick Ryder, *The geometry of rank decompositions of matrix multiplication ii: 3×3 matrices*, to appear in JPAA.
- [Bin80] D. Bini, *Relations between exact and approximate bilinear algorithms. Applications*, *Calcolo* **17** (1980), no. 1, 87–97, DOI 10.1007/BF02575865. MR605920
- [Bir46] Garrett Birkhoff, *Three observations on linear algebra* (Spanish), *Univ. Nac. Tucumán. Revista A.* **5** (1946), 147–151. MR0020547
- [BL16] Markus Bläser and Vladimir Lysikov, *On degeneration of tensors and algebras*, 41st International Symposium on Mathematical Foundations of Computer Science, LIPIcs. Leibniz Int. Proc. Inform., vol. 58, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2016, pp. Art. No. 19, 11. MR3578455
- [Blä01] Markus Bläser, *Complete problems for Valiant’s class of qp -computable families of polynomials*, *Computing and combinatorics* (Guilin, 2001), *Lecture Notes in Comput. Sci.*, vol. 2108, Springer, Berlin, 2001, pp. 1–10, DOI 10.1007/3-540-44679-6_1. MR1935355
- [Blä13] Markus Bläser, *Fast matrix multiplication*, *Graduate Surveys*, no. 5, *Theory of Computing Library*, 2013.
- [Blä14] Markus Bläser, *Explicit tensors*, *Perspectives in Computational Complexity*, Springer, 2014, pp. 117–130.
- [BLR80] Dario Bini, Grazia Lotti, and Francesco Romani, *Approximate solutions for the bilinear form computational problem*, *SIAM J. Comput.* **9** (1980), no. 4, 692–697, DOI 10.1137/0209053. MR592760
- [BOC92] Michael Ben-Or and Richard Cleve, *Computing algebraic formulas using a constant number of registers*, *SIAM J. Comput.* **21** (1992), no. 1, 54–58, DOI 10.1137/0221006. MR1148816
- [Bri87] Michel Brion, *Sur l’image de l’application moment* (French), *Séminaire d’algèbre Paul Dubreil et Marie-Paule Malliavin* (Paris, 1986), *Lecture Notes in Math.*, vol. 1296, Springer, Berlin, 1987, pp. 177–192, DOI 10.1007/BFb0078526. MR932055
- [BRR17] Emmanuel Briand, Amarpreet Rattan, and Mercedes Rosas, *On the growth of Kronecker coefficients* (English, with English and French summaries), *Sém. Lothar. Combin.* **78B** (2017), Art. 70, 12. MR3678652
- [Bur15] Vladimir P. Burichenko, *Symmetries of matrix multiplication algorithms. I*, *CoRR abs/1508.01110* (2015).
- [CFS+16] Shawn X. Cui, Michael H. Freedman, Or Sattath, Richard Stong, and Greg Minton, *Quantum max-flow/min-cut*, *J. Math. Phys.* **57** (2016), no. 6, 062206, 18, DOI 10.1063/1.4954231. MR3513725
- [CFW10] Danny Calegari, Michael H. Freedman, and Kevin Walker, *Positivity of the universal pairing in 3 dimensions*, *J. Amer. Math. Soc.* **23** (2010), no. 1, 107–188, DOI 10.1090/S0894-0347-09-00642-0. MR2552250
- [CGJ] Matthias Christandl, Fulvio Gesmundo, and Asger Kjørluff Jensen, *Border rank is not multiplicative under the tensor product*, arXiv:1801.04852.
- [CGL+18] A. Conner, F. Gesmundo, J. M. Landsberg, E. Ventura, and Y. Wang, *A geometric study of Strassen’s asymptotic rank conjecture and its variants*, *ArXiv e-prints arXiv:1811.05511* (2018).
- [CGLM08] Pierre Comon, Gene Golub, Lek-Heng Lim, and Bernard Mourrain, *Symmetric tensors and symmetric tensor rank*, *SIAM J. Matrix Anal. Appl.* **30** (2008), no. 3, 1254–1279, DOI 10.1137/060661569. MR2447451
- [CGLVa] Austin Conner, Fulvio Gesmundo, J.M. Landsberg, and Emanuele Ventura, *Kronecker powers of tensors with symmetry*, preprint.
- [CGLVb] ———, *Tensors with maximal symmetries*, preprint.
- [CHI+18] Luca Chiantini, Jonathan D. Hauenstein, Christian Ikenmeyer, Joseph M. Landsberg, and Giorgio Ottaviani, *Polynomials and the exponent of matrix multiplication*, *Bull. Lond. Math. Soc.* **50** (2018), no. 3, 369–389, DOI 10.1112/blms.12147. MR3829726
- [CHM07] Matthias Christandl, Aram W. Harrow, and Graeme Mitchison, *Nonzero Kronecker coefficients and what they tell us about spectra*, *Comm. Math. Phys.* **270** (2007), no. 3, 575–585, DOI 10.1007/s00220-006-0157-3. MR2276458

- [CHSH69] John F. Clauser, Michael A. Horne, Abner Shimony, and Richard A. Holt, *Proposed experiment to test local hidden-variable theories*, Phys. Rev. Lett. **23** (1969), 880–884.
- [CLO16] Luca Chiantini, Christian Ikenmeyer, J. M. Landsberg, and Giorgio Ottaviani, *The geometry of rank decompositions of matrix multiplication I: 2×2 matrices*, Exper. Math., to appear [abs/1610.08364](https://arxiv.org/abs/1610.08364) (2016).
- [CM06] Matthias Christandl and Graeme Mitchison, *The spectra of quantum states and the Kronecker coefficients of the symmetric group*, Comm. Math. Phys. **261** (2006), no. 3, 789–797, DOI 10.1007/s00220-005-1435-1. MR2197548
- [Con17] A. Conner, *A rank 18 Waring decomposition of $sM_{-}(3)$ with 432 symmetries*, ArXiv e-prints, to appear in Exper. Math. (2017).
- [COV14] Luca Chiantini, Giorgio Ottaviani, and Nick Vannieuwenhoven, *An algorithm for generic and low-rank specific identifiability of complex tensors*, SIAM J. Matrix Anal. Appl. **35** (2014), no. 4, 1265–1287, DOI 10.1137/140961389. MR3270978
- [CU03] H Cohn and C. Umans, *A group theoretic approach to fast matrix multiplication*, Proceedings of the 44th annual Symposium on Foundations of Computer Science (2003), no. 2, 438–449.
- [CVZ17] Matthias Christandl, Péter Vrana, and Jeroen Zuiddam, *Universal points in the asymptotic spectrum of tensors*, STOC’18—Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, ACM, New York, 2018, pp. 289–296. MR3826254
- [CW90] Don Coppersmith and Shmuel Winograd, *Matrix multiplication via arithmetic progressions*, J. Symbolic Comput. **9** (1990), no. 3, 251–280, DOI 10.1016/S0747-7171(08)80013-2. MR1056627
- [CW04] Matthias Christandl and Andreas Winter, *“Squashed entanglement”: an additive entanglement measure*, J. Math. Phys. **45** (2004), no. 3, 829–840, DOI 10.1063/1.1643788. MR2036165
- [DH02] Matthew J. Donald, Michal Horodecki, and Oliver Rudolph, *The uniqueness theorem for entanglement measures*, J. Math. Phys. **43** (2002), no. 9, 4252–4272, DOI 10.1063/1.1495917. Quantum information theory. MR1924435
- [DM18] Harm Derksen and Visu Makam, *On non-commutative rank and tensor rank*, Linear Multilinear Algebra **66** (2018), no. 6, 1069–1084, DOI 10.1080/03081087.2017.1337058. MR3781583
- [EFS56] P. Elias, A. Feinstein, and C. E. Shannon, *A note on the maximum flow through a network*, IRE. Transactions on Information Theory **2,4** (1956), 117–119.
- [EGOW17] Klim Efremenko, Ankit Garg, Rafael Oliveira, and Avi Wigderson, *Barriers for rank methods in arithmetic complexity*, 9th Innovations in Theoretical Computer Science, LIPIcs. Leibniz Int. Proc. Inform., vol. 94, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018, pp. Art. No. 1, 19. MR3761737
- [Eis13] J. Eisert, *Entanglement and tensor network states*, Autumn School on Correlated Electrons: Emergent Phenomena in Correlated Matter Jülich, Germany, 23-27. September 2013, 2013.
- [EPR35] A. Einstein, B. Podolsky, and N. Rosen, *Can quantum-mechanical description of physical reality be considered complete?*, Phys. Rev. **47** (1935), 777–780.
- [Erd42] P. Erdős, *On an elementary proof of some asymptotic formulas in the theory of partitions*, Ann. of Math. (2) **43** (1942), 437–450, DOI 10.2307/1968802. MR0006749
- [Erd47] P. Erdős, *Some remarks on the theory of graphs*, Bull. Amer. Math. Soc. **53** (1947), 292–294, DOI 10.1090/S0002-9904-1947-08785-1. MR0019911
- [FF56] L. R. Ford Jr. and D. R. Fulkerson, *Maximal flow through a network*, Canad. J. Math. **8** (1956), 399–404, DOI 10.4153/CJM-1956-045-5. MR0079251
- [FH91] William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991. A first course; Readings in Mathematics. MR1153249
- [FNW91] M. Fannes, B. Nachtergaele, and R. F. Werner, *Valence bond states on quantum spin chains as ground states with spectral gap*, J. Phys. A **24** (1991), no. 4, L185–L190. MR1104168
- [FNW92] M. Fannes, B. Nachtergaele, and R. F. Werner, *Finitely correlated states on quantum spin chains*, Comm. Math. Phys. **144** (1992), no. 3, 443–490. MR1158756

- [FR07] Marc Fortin and Christophe Reutenauer, *Commutative/noncommutative rank of linear matrices and subspaces of matrices of low rank*, *Sém. Lothar. Combin.* **52** (2004/07), Art. B52f, 12. MR2123060
- [Fra02] Matthias Franz, *Moment polytopes of projective G -varieties and tensor products of symmetric group representations*, *J. Lie Theory* **12** (2002), no. 2, 539–549. MR1923785
- [Gal17] Maciej Gałazka, *Vector bundles give equations of cactus varieties*, *Linear Algebra Appl.* **521** (2017), 254–262, DOI 10.1016/j.laa.2016.12.005. MR3611482
- [Gar82] Devra Garfinkle, *A NEW CONSTRUCTION OF THE JOSEPH IDEAL*, ProQuest LLC, Ann Arbor, MI, 1982. Thesis (Ph.D.)—Massachusetts Institute of Technology. MR2941017
- [GLW18] Fulvio Gesmundo, J. M. Landsberg, and Michael Walter, *Matrix product states and the quantum max-flow/min-cut conjectures*, *J. Math. Phys.* **59** (2018), no. 10, 102205, 11, DOI 10.1063/1.5026985. MR3865047
- [Hac12] Wolfgang Hackbusch, *Tensor spaces and numerical tensor calculus*, Springer Series in Computational Mathematics, vol. 42, Springer, Heidelberg, 2012. MR3236394
- [Har95] Joe Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1995. A first course; Corrected reprint of the 1992 original. MR1416564
- [Har01] Lucien Hardy, *Why quantum theory?*, Non-locality and modality (Cracow, 2001), NATO Sci. Ser. II Math. Phys. Chem., vol. 64, Kluwer Acad. Publ., Dordrecht, 2002, pp. 61–73. MR2031969
- [Has07] M. B. Hastings, *An area law for one-dimensional quantum systems*, *J. Stat. Mech. Theory Exp.* **8** (2007), P08024, 14, DOI 10.1088/1742-5468/2007/08/p08024. MR2338267
- [Has17] Matthew B. Hastings, *The asymptotics of quantum max-flow min-cut*, *Comm. Math. Phys.* **351** (2017), no. 1, 387–418, DOI 10.1007/s00220-016-2791-8. MR3613509
- [HIL13] Jonathan D. Hauenstein, Christian Ikenmeyer, and J. M. Landsberg, *Equations for lower bounds on border rank*, *Exp. Math.* **22** (2013), no. 4, 372–383, DOI 10.1080/10586458.2013.825892. MR3171099
- [HLP52] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge, at the University Press, 1952. 2d ed. MR0046395
- [HMT] Elitza Hristova, Tomasz Maciazek, and Valdemar V. Tsanov, *On momentum images of representations and secant varieties*, arXiv:1504.01110.
- [HR00] G. H. Hardy and S. Ramanujan, *Asymptotic formulæ in combinatory analysis [Proc. London Math. Soc. (2) 17 (1918), 75–115]*, Collected papers of Srinivasa Ramanujan, AMS Chelsea Publ., Providence, RI, 2000, pp. 276–309. MR2280879
- [IL16] Thomas A. Ivey and Joseph M. Landsberg, *Cartan for beginners*, Graduate Studies in Mathematics, vol. 175, American Mathematical Society, Providence, RI, 2016. Differential geometry via moving frames and exterior differential systems; Second edition [of MR2003610]. MR3586335
- [IQS17] Gábor Ivanyos, Youming Qiao, and K. V. Subrahmanyam, *Non-commutative Edmonds’ problem and matrix semi-invariants*, *Comput. Complexity* **26** (2017), no. 3, 717–763, DOI 10.1007/s00037-016-0143-x. MR3691737
- [Kle31] O. Klein, *Quantum coding*, *Z. Phys.* **72** (1931), 767–775.
- [Kly04] Alexander Klyachko, *Vector bundles, linear representations, and spectral problems*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), Higher Ed. Press, Beijing, 2002, pp. 599–613. MR1957068
- [KN79] George Kempf and Linda Ness, *The length of vectors in representation spaces*, Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), Lecture Notes in Math., vol. 732, Springer, Berlin, 1979, pp. 233–243. MR555701
- [Kna02] Anthony W. Knap, *Lie groups beyond an introduction*, 2nd ed., Progress in Mathematics, vol. 140, Birkhäuser Boston, Inc., Boston, MA, 2002. MR1920389
- [KSZ93] A. Klümper, A. Schadschneider, and J. Zittartz, *Matrix product ground states for one-dimensional spin-1 quantum antiferromagnets*, *EPL (Europhysics Letters)* **24** (1993), no. 4, 293.

- [KW01] M. Keyl and R. F. Werner, *Estimating the spectrum of a density operator*, Phys. Rev. A (3) **64** (2001), no. 5, 052311, 5, DOI 10.1103/PhysRevA.64.052311. MR1878924
- [Lan06] J. M. Landsberg, *The border rank of the multiplication of 2×2 matrices is seven*, J. Amer. Math. Soc. **19** (2006), no. 2, 447–459, DOI 10.1090/S0894-0347-05-00506-0. MR2188132
- [Lan12] J. M. Landsberg, *Tensors: geometry and applications*, Graduate Studies in Mathematics, vol. 128, American Mathematical Society, Providence, RI, 2012. MR2865915
- [Lan17] J. M. Landsberg, *Geometry and complexity theory*, Cambridge Studies in Advanced Mathematics, vol. 169, Cambridge University Press, Cambridge, 2017. MR3729273
- [Lei16] Arielle Leitner, *Limits under conjugacy of the diagonal subgroup in $SL_n(\mathbb{R})$* , Proc. Amer. Math. Soc. **144** (2016), no. 8, 3243–3254, DOI 10.1090/proc/12959. MR3503693
- [LG14] François Le Gall, *Powers of tensors and fast matrix multiplication*, ISSAC 2014—Proceedings of the 39th International Symposium on Symbolic and Algebraic Computation, ACM, New York, 2014, pp. 296–303, DOI 10.1145/2608628.2608664. MR3239939
- [Lic85] Thomas Lickteig, *Typical tensorial rank*, Linear Algebra Appl. **69** (1985), 95–120, DOI 10.1016/0024-3795(85)90070-9. MR798367
- [Lie73] Elliott H. Lieb, *Convex trace functions and the Wigner-Yanase-Dyson conjecture*, Advances in Math. **11** (1973), 267–288, DOI 10.1016/0001-8708(73)90011-X. MR0332080
- [LM02] Joseph M. Landsberg and Laurent Manivel, *Construction and classification of complex simple Lie algebras via projective geometry*, Selecta Math. (N.S.) **8** (2002), no. 1, 137–159, DOI 10.1007/s00029-002-8103-5. MR1890196
- [LM08] J. M. Landsberg and L. Manivel, *Generalizations of Strassen’s equations for secant varieties of Segre varieties*, Comm. Algebra **36** (2008), no. 2, 405–422, DOI 10.1080/00927870701715746. MR2387532
- [LM16] Joseph M. Landsberg and Mateusz Michałek, *A $2n^2 - \log_2(n) - 1$ lower bound for the border rank of matrix multiplication*, Int. Math. Res. Not. IMRN **15** (2018), 4722–4733, DOI 10.1093/imrn/rnx025. MR3842382
- [LM17a] J. M. Landsberg and Mateusz Michałek, *Abelian tensors* (English, with English and French summaries), J. Math. Pures Appl. (9) **108** (2017), no. 3, 333–371, DOI 10.1016/j.matpur.2016.11.004. MR3682743
- [LM17b] J. M. Landsberg and Mateusz Michałek, *On the geometry of border rank decompositions for matrix multiplication and other tensors with symmetry*, SIAM J. Appl. Algebra Geom. **1** (2017), no. 1, 2–19, DOI 10.1137/16M1067457. MR3633766
- [LO13] J. M. Landsberg and Giorgio Ottaviani, *Equations for secant varieties of Veronese and other varieties*, Ann. Mat. Pura Appl. (4) **192** (2013), no. 4, 569–606, DOI 10.1007/s10231-011-0238-6. MR3081636
- [LO15] Joseph M. Landsberg and Giorgio Ottaviani, *New lower bounds for the border rank of matrix multiplication*, Theory Comput. **11** (2015), 285–298, DOI 10.4086/toc.2015.v011a011. MR3376667
- [LQY12] Joseph M. Landsberg, Yang Qi, and Ke Ye, *On the geometry of tensor network states*, Quantum Inf. Comput. **12** (2012), no. 3-4, 346–354. MR2933533
- [LR73] Elliott H. Lieb and Mary Beth Ruskai, *Proof of the strong subadditivity of quantum-mechanical entropy*, J. Mathematical Phys. **14** (1973), 1938–1941, DOI 10.1063/1.1666274. With an appendix by B. Simon. MR0345558
- [Mac95] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995. With contributions by A. Zelevinsky; Oxford Science Publications. MR1354144
- [Man15] Laurent Manivel, *On the asymptotics of Kronecker coefficients*, J. Algebraic Combin. **42** (2015), no. 4, 999–1025, DOI 10.1007/s10801-015-0614-1. MR3417256
- [MS] Mateusz Michałek and Yaroslav Shitov, *Quantum version of weilandt’s inequality revisited*, arXiv:1809.04387.

- [MSV] Mateusz Michalek, Tim Seynnaeve, and Frank Verstraete, *A higher dimensional version of the quantum weilandt's theorem*, arXiv:1811.05502.
- [Mum95] David Mumford, *Algebraic geometry. I*, Classics in Mathematics, Springer-Verlag, Berlin, 1995. Complex projective varieties; Reprint of the 1976 edition. MR1344216
- [Nes84] Linda Ness, *A stratification of the null cone via the moment map*, Amer. J. Math. **106** (1984), no. 6, 1281–1329, DOI 10.2307/2374395. With an appendix by David Mumford. MR765581
- [Nie99] M. A. Nielsen, *Conditions for a class of entanglement transformations*, P H Y S I C A L R E V I E W L E T T E R S **83** (1999), 436–439.
- [Pan66] V. Ja. Pan, *On means of calculating values of polynomials* (Russian), Uspehi Mat. Nauk **21** (1966), no. 1 (127), 103–134. MR0207178
- [Paz84] Azaria Paz, *An application of the Cayley-Hamilton theorem to matrix polynomials in several variables*, Linear and Multilinear Algebra **15** (1984), no. 2, 161–170, DOI 10.1080/03081088408817585. MR740668
- [PGVWC07] D. Perez-Garcia, F. Verstraete, M. M. Wolf, and J. I. Cirac, *Matrix product state representations*, Quantum Inf. Comput. **7** (2007), no. 5–6, 401–430. MR2347213
- [Pro07] Claudio Procesi, *Lie groups*, Universitext, Springer, New York, 2007. An approach through invariants and representations. MR2265844
- [Sar] Parth Sarin, *On the geometry of tensor network states of $2 \times n$ grids*, preprint arXiv, to appear in LAA.
- [Sch81] A. Schönhage, *Partial and total matrix multiplication*, SIAM J. Comput. **10** (1981), no. 3, 434–455, DOI 10.1137/0210032. MR623057
- [Sch95] Benjamin Schumacher, *Quantum coding*, Phys. Rev. A (3) **51** (1995), no. 4, 2738–2747, DOI 10.1103/PhysRevA.51.2738. MR1328824
- [Sch03] Rüdiger Schack, *Quantum theory from four of Hardy's axioms*, Found. Phys. **33** (2003), no. 10, 1461–1468, DOI 10.1023/A:1026044329659. Special issue dedicated to David Mermin, Part I. MR2039620
- [Sei] A. Seigal, *Ranks and symmetric ranks of cubic surfaces*, preprint, arXiv:1801.05377.
- [Sha48] C. E. Shannon, *A mathematical theory of communication*, Bell System Tech. J. **27** (1948), 379–423, 623–656, DOI 10.1002/j.1538-7305.1948.tb01338.x. MR0026286
- [Sha13] Igor R. Shafarevich, *Basic algebraic geometry. 2*, 3rd ed., Springer, Heidelberg, 2013. Schemes and complex manifolds; Translated from the 2007 third Russian edition by Miles Reid. MR3100288
- [Shi17] Yaroslav Shitov, *A counterexample to Comon's conjecture*, SIAM J. Appl. Algebra Geom. **2** (2018), no. 3, 428–443, DOI 10.1137/17M1131970. MR3852707
- [Shi18] Yaroslav Shitov, *A counterexample to Comon's conjecture*, SIAM J. Appl. Algebra Geom. **2** (2018), no. 3, 428–443, DOI 10.1137/17M1131970. MR3852707
- [Smi13] A. V. Smirnov, *The bilinear complexity and practical algorithms for matrix multiplication* (Russian, with Russian summary), Zh. Vychisl. Mat. Mat. Fiz. **53** (2013), no. 12, 1970–1984, DOI 10.1134/S0965542513120129; English transl., Comput. Math. Math. Phys. **53** (2013), no. 12, 1781–1795. MR3146566
- [Smi14] A. V. Smirnov, *A bilinear algorithm of length 22 for approximate multiplication of 2×7 and 7×2 matrices*, Comput. Math. Math. Phys. **55** (2015), no. 4, 541–545, DOI 10.1134/S0965542515040168. MR3343116
- [SPGaWC10] Mikel Sanz, David Pérez-García, Michael M. Wolf, and Juan I. Cirac, *A quantum version of Wielandt's inequality*, IEEE Trans. Inform. Theory **56** (2010), no. 9, 4668–4673, DOI 10.1109/TIT.2010.2054552. MR2807352
- [ST16] W. F. Sawin and T. Tao, *Notes on the “slice rank” of tensors*, Available at <https://terrytao.wordpress.com/2016/08/24/notes-on-the-slice-rank-of-tensors/>.
- [Sto] A. Stothers, *On the complexity of matrix multiplication*, PhD thesis, University of Edinburgh, 2010.
- [Str69] Volker Strassen, *Gaussian elimination is not optimal*, Numer. Math. **13** (1969), 354–356, DOI 10.1007/BF02165411. MR0248973
- [Str73] Volker Strassen, *Vermeidung von Divisionen* (German, with English summary), J. Reine Angew. Math. **264** (1973), 184–202. MR0521168
- [Str83] V. Strassen, *Rank and optimal computation of generic tensors*, Linear Algebra Appl. **52/53** (1983), 645–685, DOI 10.1016/0024-3795(83)80041-X. MR709378

- [Str87] V. Strassen, *Relative bilinear complexity and matrix multiplication*, J. Reine Angew. Math. **375/376** (1987), 406–443, DOI 10.1515/crll.1987.375-376.406. MR882307
- [Str91] V. Strassen, *Degeneration and complexity of bilinear maps: some asymptotic spectra*, J. Reine Angew. Math. **413** (1991), 127–180, DOI 10.1515/crll.1991.413.127. MR1089800
- [Str94] V. Strassen, *Algebra and complexity*, First European Congress of Mathematics, Vol. II (Paris, 1992), Progr. Math., vol. 120, Birkhäuser, Basel, 1994, pp. 429–446, DOI 10.1007/s10107-008-0221-1. MR1341854
- [Str05] Volker Strassen, *Komplexität und Geometrie bilinearer Abbildungen* (German, with English summary), Jahresber. Deutsch. Math.-Verein. **107** (2005), no. 1, 3–31. MR2138544
- [Sur00] B. Sury, *An elementary proof of the Hilbert-Mumford criterion*, Electron. J. Linear Algebra **7** (2000), 174–177, DOI 10.13001/1081-3810.1053. MR1781469
- [Tob91] V. Tobler, *Spezialisierung und degeneration von tensoren*, PhD thesis (1991).
- [Tsa] Valdemar V. Tsanov, *Secant varieties and degrees of invariants*, arXiv:1811.12048.
- [VC04] F. Verstraete and J. I. Cirac, *Renormalization algorithms for Quantum-Many Body Systems in two and higher dimensions*, eprint arXiv:cond-mat/0407066 (2004).
- [Wie50] Helmut Wielandt, *Unzerlegbare, nicht negative Matrizen* (German), Math. Z. **52** (1950), 642–648, DOI 10.1007/BF02230720. MR0035265
- [Wil] Virginia Williams, *Breaking the coppersmith-winograd barrier*, preprint.
- [Wil92] N. J. Wildberger, *The moment map of a Lie group representation*, Trans. Amer. Math. Soc. **330** (1992), no. 1, 257–268, DOI 10.2307/2154163. MR1040046

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- 74 **Lawrence C. Evans**, Weak Convergence Methods for Nonlinear Partial Differential Equations, 1990

Tensors are used throughout the sciences, especially in solid state physics and quantum information theory. This book brings a geometric perspective to the use of tensors in these areas. It begins with an introduction to the geometry of tensors and provides geometric expositions of the basics of quantum information theory, Strassen's laser method for matrix multiplication, and moment maps in algebraic geometry. It also details several exciting recent developments regarding tensors in general. In particular, it discusses and explains the following material previously only available in the original research papers: (1) Shitov's 2017 refutation of longstanding conjectures of Strassen on rank additivity and Comon on symmetric rank; (2) The 2017 Christandl-Vrana-Zuiddam quantum spectral points that bring together quantum information theory, the asymptotic geometry of tensors, matrix multiplication complexity, and moment polytopes in geometric invariant theory; (3) the use of representation theory in quantum information theory, including the solution of the quantum marginal problem; (4) the use of tensor network states in solid state physics, and (5) recent geometric paths towards upper bounds for the complexity of matrix multiplication. Numerous open problems appropriate for graduate students and post-docs are included throughout.



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ISBN: 978-1-4704-5136-3



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