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Preface

This book presents and expands upon material presented at a July 2017 CBMS lecture series on the geometry of tensors and applications. The expansion includes two very exciting recent developments regarding tensors:

- Y. Shitov refuted two longstanding conjectures; V. Strassen's 1973 additivity conjecture and P. Comon's symmetric rank conjecture. I present a detailed exposition of his refutation of Strassen's conjecture after giving a general introduction to Shitov's perspective that allowed resolution of these conjectures.
- Christandl-Vrana-Zuiddam introduced *quantum spectral points*, which brought together Strassen's spectral theory (developed for the study of matrix multiplication) and quantum information theory, by employing representation theory and rational moment polytopes. This work brings together many interesting ideas and opens doors to future research. I give a detailed exposition of their theory.

In addition to these two advances, I also present the following recent developments:

- In my lectures I had included what at the time were several open questions regarding tensor network states, which I thought should be resolvable with geometric techniques. My impression turned out to be too correct- as I started writing up these notes, two of them were essentially solved [**MS**, **MSV**]. I present expositions of both, as well as an introduction to tensor network states with an emphasis on applications to solid state physics.
- An exposition of the elementary proofs in [**DM18**] and [**EGOW17**] of the limits of rank methods in proving border rank lower bounds.
- Recent approaches towards bounding the exponent of matrix multiplication that include: a description of the use of the symmetrized matrix multiplication polynomial to bound the exponent of matrix multiplication [**CHI+18**, **Con17**], the potential use symmetry groups of tensors in the laser method [**CGLVb**], and an exploration of Strassen's generalization of the conjecture that $\omega = 2$ to all tight tensors [**CGL+18**].

Beyond the recent developments, these notes include expositions of Strassen's laser method, Strassen's spectral theory, basics of quantum information theory, the resolution of the quantum marginal problem by Klyatchko and independently Christandl-Mitchison, and a largely self-contained exposition of moment polytopes in projective algebraic geometry as developed by Brion.

The book is divided into three parts.

- The first part is a brief introduction to tensors, complementary to the one in [**Lan12**].
- The second part is a study of tensors by employing linear algebra. The topics covered are as follows:

  The use of determinantal equations for bounding border rank. In the past few years, severe limits on determinantal methods were discovered. I present elementary limitations due to Derksen-Makam and Efremenko-Garg-Oliveira-Wigderson that complement the more subtle and geometric limitations discovered by Buczynski-Galzacka (following Bernardi-Ranestad) detailed in [**Lan17**, §10.2].

  Shitov's refutations discussed above appear in this part.

  The remainder of this part is a discussion of tensor networks with an emphasis on their use in condensed matter theory. To quote [**PGVWC07**] Entanglement "is a blessing for *quantum information theory* - it facilitates exponential speed-ups in quantum simulation and quantum computing – it is often more a curse for *condensed matter theory* where the complexity of such systems make them hardly tractable by classical means." For condensed matter theory and other areas discussed in this chapter that are haunted by the so-called "curse of dimensionality", the problems are approached by reducing to linear algebra via tensor networks. This section also includes pointers to the physics literature. I warn mathematicians that the referenced papers do not always define their terms, as is considered mandatory in mathematics.
- The third part discusses the asymptotic geometry of tensors. This theory is based on Shannon's information theory and probability. I present the requisite background material in Chapter 4. Then, in Chapter 5, I explain how ideas from information theory were used to obtain upper bounds on the exponent of matrix multiplication, and led to Strassen's asymptotic spectra. In order to discuss the quantum spectral points in context, in Chapter 6 I give a very brief introduction to quantum information theory. In reference to the quote from [**PGVWC07**] above, for quantum information theory, representation theory is used to extract qualitative information about tensors in the above-mentioned "cursed" spaces. The topics of Chapters 4, 5 and 6 are brought together in Chapter 7 with the quantum spectral points. The main result regarding the quantum spectral points depends heavily on a very special case of Brion's theorem on the rational moment polytope. I give a proof of Brion's theorem in general, as well as a general introduction to moment maps and moment polytopes in algebraic geometry in Chapter 8.

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