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Lectures on Field Theory and Topology

Daniel S. Freed



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To {Alex, David, Sonia},
with love and gratitude

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Preface

In early August of 2017 David Ayala and Ryan Grady organized a CBMS conference on *Topological and Geometric Methods in Quantum Field Theory* at Montana State University in Bozeman. I gave a series of ten lectures which form the basis for this volume. There were supplementary lectures by several other mathematicians and physicists. Some of those are collected in a companion volume [AFG]. Many students and postdocs attended, as did several more senior mathematicians. The beautiful natural setting and relaxed atmosphere were perfect catalysts for interactions among the participants.

I thank David and Ryan for their outstanding work bringing the conference together. Agnes Beaudry, Robert Bryant, Tudor Dimofte, Sam Gunningham, Max Metlitski, Dave Morrison, and Andy Neitzke gave terrific lectures at the conference. Beth Burroughs, Jane Crawford, and Katie Sutich in the Department of Mathematical Sciences at MSU provided massive assistance during the planning and execution of the conference. The College of Letters and Science and the VP of Research at Montana State University provided support for the CBMS conference as well. The Aspen Center for Physics provided me hospitality during the period when I wrote the first draft of the lectures. I thank them all.

Dan Freed
December, 2018

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