NAVIER-STOKES EQUATIONS

THEORY AND NUMERICAL ANALYSIS

ROGER TEMAM

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American Mathematical Society • Providence, Rhode Island

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Dedicated with deep respect to the memory of Jean Leray

Contents

Note to the reader: Two distinct bibliographies are available. The bibliography that is original to this volume appears at the end of the book, and its references are made by names followed by a one- or two-digit number in brackets. The extended bibliography to Appendix III appears within that appendix, and its references are made by the names of the authors followed by the year of publication between parentheses.

Preface to the AMS Chelsea edition

This edition reproduces the book initially published in 1977 by North-Holland. In its presentation, it has been fully retypeset by AMS. In its content, except for some minor editing, it is identical to the third revised version published in 1984. It is likely that, if written now, the book would be different in several respects. On the other hand, introducing changes in this new edition would have required extensive work with doubtful results and a high probability of introducing new errors. Hence it has been decided to reproduce the book as it was in its last edition.

The new material in this book is Appendix III, reproducing a survey article which first appeared in a volume published by Birkhäuser. This appendix contains a few aspects not addressed in the earlier edition, in particular: a short derivation of the Navier-Stokes equations from the basic conservation principles in continuum mechanics, some further historical perspectives, and some indications on new developments. It also surveys some aspects of related equations which are not the purpose of the book: the Euler equations and the compressible Navier-Stokes equations. It is suggested to the reader to peruse this appendix before reading the core of the book.

If the book were to be written or rewritten now, the following difficulty would have to be addressed: in the writing of the first edition, it was attempted, to some extent, to include all the material available on the existence and uniqueness of solutions for the Navier-Stokes equations and their approximation. The body of knowledge has considerably expanded since then, and now a single book could not comprehend all this material; hence choices would have to be made. As we say elsewhere the numerical aspects have expanded into a field of their own, Computational Fluid Dynamics. On the theoretical side, there are a large number of new developments which are described in Appendix III. Let us mention here some of these developments which are close to this volume. New simpler proofs were derived for technical results very often used in this book (see e.g. the footnote before Proposition 1.1.1, Remark 1.2.7 and Remark 2.1.6 iii). The space-periodic case has been very much studied: it is conceptually simpler and Fourier series can be used, but many of the difficulties are the same as for the no-slip case studied here. The main simplifications are due to the absence of the difficulties related to the boundary layer (another subject under development at this time, absent from this book). New results on time and space analyticity were proven (analyticity in time and Gevrey regularity in space). Although results of analyticity were available at the time of the writing of this book, the proofs of the new results are much closer to the spirit of this book. Substantial developments occured also on the large time behavior of the solutions to the Navier-Stokes equations and the relation with turbulence theory. Most of these new results not developed in this book are available in the lecture notes of R. Temam (1995) and in the forthcoming book by C. Foias,

O. Manley, R. Rosa and R. Temam (2001) which serve as possible continuations of this book. Finally the control of turbulent flows is another subject under development which became accessible and which is not present in this book, except for some remarks at the end of Appendix III, with two figures representing the results of extensive numerical simulations.

I am very pleased that the American Mathematical Society decided to republish this book and I hope this new edition will be useful. I would like to thank especially Susan Friedlander who initiated this project and Sergei Gelfand who very effectively managed it. I would also like to thank a number of young colleagues who helped me read (once more!) this book, and made a number of corrections and remarks, namely Didier Bresch, Brian Ewald, Olivier Goubet, Changbing Hu, François Jauberteau, Jean-Michel Rakotoson, Jie Shen, Shouhong Wang, Xiaoming Wang, and Mohammed Ziane.

As evidenced by the numerous references to his work, this book has been very much influenced by what I learned from my teacher Jacques-Louis Lions. Further back in the history of the Navier-Stokes equations, we owe to Jean Leray (1906-1998) considerable pioneering work on the theory of the Navier-Stokes equations (see the Introduction to Appendix III). He has also done considerable pioneering work in several other areas of mathematics. In his collected works published in 1999, and elsewhere, he is recognized as one of the most prominent mathematicians of the twentieth century.

It was given to me to speak at Jean Leray's seminar at the Collège de France in Paris, or simply to attend it, in the ancient "Salle 5" full of history: it was always a humbling and unforgettable experience for a young researcher. In grateful reminiscence of the kind support and attention that he devoted to the young researcher that I was when I wrote this book, I dedicate this new edition, with deep respect, to his memory.

September 2000

Preface to the third (revised) edition

Since the publication of this book, numerous articles have appeared, connected with the theory of the numerical approximation of the Navier–Stokes equations. The increasing interest for these equations is due in part to the important role that they play in many scientific and industrial applications of current interest like aeronautical sciences, meteorology, thermo-hydraulics, petroleum industry, plasma physics, etc... It is also due to the development of the computing power which is now available with the new computers and the computing power which we can foresee for a near future with supercomputers. The process of solving problems in fluid dynamics numerically on a computer is called Computational Fluid Dynamics (CFD). This subject has considerably expanded in recent years; there are now thousands of researchers, many applications, and an enormous literature in CFD, and the expansion will likely continue.

This present book stands at the boundary between computational fluid dynamics and mathematical analysis to which CFD is firmly tied. Even if we restrict ourselves to the theory and numerical analysis of the Navier–Stokes equations for incompressible fluids, the rapid expansion of these subjects make it now impossible to include in a single volume a comprehensive presentation of them. However, we have though that the basic questions studied in this volume will be of interest for some time and that the book, in its present form remains useful. For the readers interested in the most recent developments or more specialized ones. this new edition contains a revision and an updating of the bibliography. It contains also (in the Additional comments to the revised edition, p. 381) a description, necessarily uncomplete, of the directions in which progresses have been made recently.

Paris, January 1984

Foreword

In the present work we derive a number of results concerned with the theory and numerical analysis of the Navier–Stokes equations for viscous incompressible fluids. We shall deal with the following problems: on the one hand, a description of the known results on the existence, the uniqueness and in a few cases the regularity of solutions in the linear and non-linear cases, the steady and time-dependent cases; on the other hand, the approximation of these problems by discretization: finite difference and finite element methods for the space variables, finite differences and fractional steps for the time variable. The questions of stability and convergence of the numerical procedures are treated as fully as possible. We shall not restrict ourselves to these theoretical aspects: in particular, in the Appendix we give details of how to program one of the methods. All the methods we study have in fact been applied, but it has not been possible to present details of the effective implementation of all the methods. The theoretical results that we present (existence, uniqueness,...) are only very basic results and none of them is new; however we have tried as far as possible to give a simple and self-contained treatment. Energy and compactness methods lie at the very heart of the two types of problems we have gone into, and they form the natural link between them.

Let us give a more detailed description of the contents of this work: we consider first the linearized stationary case (Chapter 1), then the non-linear stationary case (Chapter 2), and finally the full non-linear time-dependent case (Chapter 3). At each stage we introduce new mathematical tools, useful both in themselves and in readiness for subsequent steps.

In Chapter 1, after a brief presentation of results on existence and uniqueness, we describe the approximation of the Stokes problem by various finite-difference and finite-element methods. This gives us an opportunity to introduce various methods of approximation of the divergence-free vector functions which are also vital for the numerical aspects of the problems studied in Chapters 2 and 3.

In Chapter 2 we introduce results on compactness in both the continuous and the discrete cases. We then extend the results obtained for the linear case in the preceding chapter to the non-linear case. The chapter ends with a proof of the non-uniqueness of solutions of the stationary Navier–Stokes equations, obtained by bifurcation and topological methods. The presentation is essentially self-contained.

Chapter 3 deals with the full non-linear time-dependent case. We first present a few results typical of the the present state of the mathematical theory of the Navier–Stokes equations (existence and uniqueness theorems). We then present a brief introduction to the numerical aspects of the problem, combining the discretization of the space variables discussed in Chapter 1 with the usual methods of discretization for the time variable. The stability and convergence problems are treated by energy methods. We also consider the fractional step method and the method of artificial compressibility.

This brief description of the contents will suffice to show that this book is in no sense a systematic study of the subject. Many aspects of the Navier–Stokes equations are not touched on here. Several interesting approaches to the existence and uniqueness problems, such as semi-groups, singular integral operators and Riemannian manifold methods, are omitted. As for the numerical aspects of the problem, we have not considered the particle approach nor the related methods developed by the Los Alamos Laboratory.

We have, moreover, restricted ourselves severely to the Navier–Stokes equations; a whole range of problems which can be treated by the same methods are not covered here; nor are the difficult problems of turbulence and high Reynolds number flows.

The material covered by this book was taught at the University of Maryland in the first semester of 1972–3 as a part of a special year on the Navier–Stokes equations and non-linear partial differential equations. The corresponding lecture notes published by the University of Maryland constitute the first version of this book.

I am extremely grateful to my colleagues in the Department of Mathematics and in the Institute of Fluid Dynamics and Applied Mathematics at the University of Maryland for the interest they showed in the elaboration of the notes. Direct contributions to the preparation of the manuscript were made by Arlett Williamson, and by Professors J. Osborn, J. Sather and P. Wolfe. I should like to thank them for correcting some of my mistakes in English and for their interesting comments and suggestions, all of which helped to improve the manuscript. Useful points were also made by Mrs Pelissier and by Messrs Fortin and Thomasset. Finally, I should like to express my thanks to the secretaries of the Mathematic Departments at Maryland and Orsay for all their assistance in the preparation of the manuscript.

Roger Temam

Comments

Chapter 1

Section I contains a preliminary study of the basic spaces V and H: the trace theorem is proved by the methods of J.L. Lions and E. Magenes, see ref. [**1**]. The characterization of H^1 given here is based on a theorem of G. de Rham of the currents theory. A more elementary proof is given in O.A. Ladyzhenskaya [**1**] for $n = 3$. A simplyfied version of O.A. Ladyzhenskaya's proof valid for all dimensions, was given in R. Temam [**9**]. Remark 1.9 gives another way for avoiding de Rham's theorem; see also the end of the footnote before Proposition 1.1.

We have not given any systematic study nor review concerning the Sobolev spaces. We restricted ourselves to recalling properties of theses spaces when needed (Section 1.1 of Chapter 1 and 2 in particular). As mentioned in the text, the reader is referred for proofs and further material to R.S. Adams [**1**], S. Agmon [**1**], J.L. Lions [**1**], J.L. Lions and E. Magenes [**1**], J. Nečas [**1**], L. Sobolev [**1**], and others.

The variational formulation of Stokes equation was first introduced (in the general frame of the non-linear case) by J. Leray [**1, 2, 3**], for the study of weak or turbulent solution of the Navier–Stokes equations. The existence of a solution of the Stokes variational problem is easily obtained by the classical Projection Theorem, whose proof is recalled for the sake of completeness. The study of the non-variational Stokes problem, and the regularity of solutions is based on the paper of L. Cattabriga [**1**] (if $n = 3$) and on the paper of S. Agmon, A. Douglis and L. Nirenberg [**1**] on elliptic systems (any dimension); these results are recalled without proofs. For another approach to the regularity cf. V.A. Solonnikov and V.E. Scadilov [**1**]. See also V.A. Solonnikov [**4**], I.I. Vorovich and V.I. Yudovich [**1**].

The concept of approximation of a normed space and of a variational problem was studied in particular by J.P. Aubin [**1**] and J. Cea [**1**]; the presentation followed here is that of R. Temam [**8**]. The discrete Poincaré Inequality (Section 3.3) and the approximation of V by finite differences are in J. Cea [**1**]. The approximation of V by conforming finite elements was first studied and used by M. Fortin [**2**]; our description of the approximations (APX2), (APX3) (conforming finite elements), follows essentially M. Fortin [**2**]. In this reference one can also find many results of computations using this type of discretization. The idea of using the bulb function is due to P.A. Raviart; the presentation of the approximation (APX2′) given here is new. The approximation (APX4) has been studied and used by J.P. Thomasset [**1**]. The material related to the non-conforming finite elements for the approximation of divergence free vector functions is due to M. Crouzeix, R. Glowinski, P.A. Raviart,

Note to the reader: These comments are those of the initial edition of the book (1977). More recent comments appear on page 337 and in Appendix III.

and the author. Other aspects of the subject (non-conforming finite elements of higher degree and more refined error estimates) can be found in M. Crouzeix and P.A. Raviart [**1**]; for numerical experiment, see F. Thomasset [**2**] and also P. Lailly [**1**] in the case of an axisymmetric three-dimensional flow.

For other applications of finite elements in fluid mechanics, see J.T. Oden, O.C. Zienkiewicz, R.H. Gallagher and T.D. Taylor [**1**], and the proceedings of the conference held in Italy, June 1976 (to appear). Concerning the general theory of finite elements, let us mention the synthesis works of I. Babuska and A.K. Aziz [**1**], P.G. Ciarlet [**1**], P.A. Raviart [**2**], G. Strang and G. Fix [**1**], and the proceedings edited by A.K. Aziz [**1**]. For more references on finite elements (in general situations) the reader is referred to the bibliography of these works. The description of finite elements methods given here is almost completely self-contained: we only assume a few specific results whose proofs would necessitate the introduction of tools quite remote from our scope.

After discretization of the Stokes problem, we have to solve a finite-dimensional linear problem where the unknown is an element $\boldsymbol{u}_h$ of a finite-dimensional space V_h. There are two possibilities:

(a) either this space $\boldsymbol{V}_h$ possesses a natural and simple basis, such that the problem is reduced to a linear system with a sparse matrix for the components of $\boldsymbol{u}_h$ in this basis; in this case we solve the problem by resolution of this linear system;
(b) or, if not, the finite-dimensional problem is not so simple to solve (ill-conditioned or non-sparse matrix), even if it possesses a unique solution. In this case, appropriate algorithms must be introduced in order to solve these problems; this is the purpose of Section 5.

The algorithms described in Section 5 were introduced in the frame of optimization theory and economics in K.J. Arrow, L. Hurwicz and H. Uzawa [**1**]; the application of these procedures to problems of hydrodynamics is studied in J. Céa, R. Glowinski and J.C. Nedelec [**1**], M. Fortin [**2**], M. Fortin, R. Peyret, and R. Temam [**1**]. See in D. Bégis [**1**], M. Fortin [**2**], and experimental investigation of the optimal choice of the parameter ϱ (or ϱ and α); a theoretical resolution of this problem in a very particular case is given in Crouzeix [**2**].

The approximation of incompressible fluids by the penalty method was first studied in R. Temam [**2a, 2b**]. The full asymptotic development of $\boldsymbol{u}_\epsilon$ given here is due to M.C. Pelissier [**1**].

Chapter 2

Section 1 develops a few standard results concerning the existence and uniqueness of solution of the nonlinear stationary Navier–Stokes equations. We follow essentially O.A. Ladyzhenskaya [**1**] and J.L. Lions [**2**]. A more complete discussion of the regularity of solutions and of the theory of hydrodynamical potentials can be found in O.A. Ladyzhenskaya [**1**]; for regularity, see also H. Fujita [**1**]. The stationary Navier–Stokes equations in an unbounded domain have been studied by R. Finn [**1**]–[**5**], R. Finn and D.R. Smith [**1, 2**], and J.G. Heywood [**1, 3**].

Some recent theoretical results concerning the stationary Navier–Stokes equations are given in C. Foias and R. Temam [**2, 3**], C. Foias and J.C. Saut [**2**], J.C. Saut and R. Temam [**2**], D. Serre [**1, 2, 3**], R. Temam [**11, 16**].

Section 2 gives discrete Sobolev inequalities and compactness theorem, whose proofs are very technical. The principle of the proofs in the case of finite-differences parallels the corresponding proofs in the continuous case (see, for instance, J.L. Lions [**1**], J.L. Lions–E. Magenes [**1**]). The proof of the discrete Sobolev inequalities has not been published before, the proof of the discrete compactness theorem can be found in P.A. Raviart [**1**]. For conforming finite elements the proofs are much simpler: in particular, for discrete compactness theorem, the problem is reduced by a simple device to the continuous case. For non-conforming finite elements the proof of the Sobolev inequality is based on specific techniques of non-conforming finite element theory. The discrete compactness theorem is proved by comparison between conforming and non-conforming elements: these results are new.

The discussion of the discretization of the stationary Navier–Stokes equations follows the principles developed in Chapter 1. The general convergence theorem is similar to that of Chapter 1 and the same types of discretization of V are considered; differences lie in the lack of uniqueness of solutions of the exact problem. The numerical algorithms of Section 3.3 have been introduced and tested in M. Fortin, R. Peyret, and R. Temam [**1**]. The modification of the trilinear form b (Chapter 2, (3.23)) corresponds to the introduction of the stabilizing term $\frac{1}{2}(\operatorname{div}\boldsymbol{u})\boldsymbol{u}$ and its discrete analog when the functions are not solenoidal; this modification was introduced and used in R. Temam [**2a, 2b, 3, 4**].

The non-uniqueness of stationary solutions of the Navier–Stokes and related equations has been investigated in recent years. The main results in this direction are due to P.H. Rabinowitz [**2**] and W. Velte [**1, 2**]. In [**2**] Rabinowitz establishes the non-uniqueness of solutions of the convection problem by explicitly constructing two different solutions (the first is the trivial one when the fluid is at rest, the second is constructed by an iterative procedure). The work of W. Velte is based on topological methods, the bifurcation theory and the topological degree theory; the problem considered in [**1**] is the convection problem as in P.H. Rabinowitz [**2**]. In [**2**], W. Velte proves the non-uniqueness of solution of the Taylor problem and the situation is very similar to the problem for which existence is proved in Section 1, although not identical. Section 4 follows closely this presentation. For other applications of bifurcation theory see in particular, J.B Keller and S. Antman [**1**], L. Nirenberg [**1**], P.H. Rabinowitz [**4, 6**] and volume 3, number 2 of the Rocky Mountain J. of Math (1973).

Chapter 3

The existence and uniqueness results for the linearized Navier–Stokes equations (Section 1) are a special case of general result of existence and uniqueness of solution of linear variational equations (see for instance, J.L. Lions–E. Magenes [**1**, vol. 2]). For completeness we have given an elementary proof of some technical results, which are usually established as easy consequences of deeper results [i.e., Lemma 1.1 which is more natural in the frame of vector valued distribution theory (L. Schwartz [**2**]) or Lemma 1.2 which can be proved by interpolation methods (J.L. Lions–E. Magenes [**1**])].

Theorem 2.1 is one of the standard compactness theorems used in the theory of nonlinear evolution equations. Other compactness theorems are proved and used in J.L. Lions [**2**]. A recent generalization of these result can be found in R. Temam [**16**].

The existence and uniqueness results related to the non-linear Navier–Stokes equations and given in Sections 3 and 4 are now classical and prolong the early works of J. Leray [**1, 2, 3**]; see E. Höpf [**1, 2**], O.A. Ladyzhenskaya [**1**], J.L. Lions [**2, 3**], J.L. Lions and G. Prodi [**1**], and J. Serrin [**3**]. Further results on the regularity of solutions and the study of the existence of classically differentiable solutions of the Navier–Stokes equations can be found in the second edition of O.A. Ladyzhenskaya [**1**]. For the analyticity of the solutions see C. Foias and G. Prodi [**1**], H. Fujita and K. Masuda [**1**], C. Kahane [**1**], K. Masuda [**1**], J. Serrin [**3**], C. Foias and R. Temam [**4**].

Let us mention also two completely different approaches to the existence and uniqueness theory that we did not treat here. The first one is that of E.B. Fabes, B.F. Jones, and N.M. Riviere [**1**] based on singular integral operator methods and giving existence and uniqueness results in L^p spaces. The other one is the method of V. Arnold [**1**] and D.G. Ebin and J. Marsden [**1**] connecting the Navier–Stokes initial value problem with the geodesics of a Riemann manifold and thus using the methods of global analysis.

The material of Section 5 containing a discussion of the stability and convergence of simple discretization schemes for the Navier–Stokes equation is essentially new; a similar study for different equations or different schemes was presented in R. Temam [**2a, 2b, 3, 4**]. Stability and convergence of some unconditionally stable one step schemes are given in O.A. Ladyzhenskaya [**5**]; for fractional step schemes see also A.J. Chorin [**2**], O.A. Ladyzhenskaya and V.I. Rivkind [**1**]. In all these references except in A.J. Chorin [**2**] the convergence is proved, as here, by obtaining appropriate *a priori* estimates of the approximated solutions and the utilization of a compactness theorem; in [**2**] A.J. Chorin assumes the existence of a very smooth solution and compares the approximated and exact solutions.

Section 7.1 is essentially an introduction to Section 7.2. The fractional step scheme described in Section 7.2 (the Projection Method) was independently introduced by A.J.Chorin [**1, 2, 3**] and the author R. Temam [**3**]; A.J. Chorin considers a slightly different form of the scheme, without the stabilizing term $\frac{1}{2}(\operatorname{div}\boldsymbol{u})\boldsymbol{u}$ (i.e., without replacing b by $\hat{b}$). Applications and other aspects of this scheme are developed in particular in C.K. Chu and G. Johansson [**1**], C.K. Chu, K.V. Morton and K.V. Roberts [**1**], M. Fortin, R. Peyret and R. Temam [**1**], M. Fortin [**1**], M. Fortin and R. Temam [**1**], G. Marshall [**1, 2**] and C.S. Peskin [**1**]. This scheme is a generalization of the fractional step method introduced and studied by G.I. Marchuk [**1**] and N.N. Yanenko [**1**] (see Section 8).

The approximation of the Navier–Stokes equations by the equations of slightly compressible fluids (Subsection 8.1) was introduced independently by A.J. Chorin [**1**] and R. Temam [**3**]. In [**1**], N.N. Yanenko considers slightly more complicated perturbed equations. The introduction of these perturbations permits the utilization of the fractional step method which is studied in Subsection 8.2. Let us point out that the schemes of Section 7 are fractional step schemes not needing the consideration of perturbed equations.

The proof of convergence of the fractional step scheme which is given here is due to R. Temam [**3, 4**] and follows the method introduced in R. Temam [**1**]. For other aspects of the Fractional Step Method, see G.I. Marchuk [**1**], N.N. Yanenko [**1, 2**] and their bibliographies; see also R. Temam [**1, 6, 7**]. Other types of perturbed problems, whose purpose is to overcome the difficulties of the constraint "div $\boldsymbol{u} = 0$" (but not to apply fractional step methods) are studied in J.L. Lions [**4**] and R. Temam [**2a, 2b**]. For the alternating direction methods and further results on fractional step methods, see O.A. Ladyzhenskaya and V.I. Rivkind [**1**], V.I. Rivkind and B.S. Epstein [**1**], and B.S. Epstein [**1**].

The material of Section 5 to 8 is only a very small part of a considerable amount of work on the approximation of fluid mechanic equations; up-to-date results and very useful references can be found in the proceeding edited by O.M. Belotserkovskii [**1**], M. Holt [**2**], H. Cabannes and R. Temam [**1**], R.D. Richtmyer [**1**], F. Thomasset [**1**] and T. Kawai [**1**]. See also the list of the references compiled by the Los Alamos Scientific Laboratory.

Many other problems can be handled by the methods used here. For the Navier–Stokes equations properly speaking one can consider different boundary conditions (see Iooss [**1**]), or periodic solutions (G. Prouse [**1, 2**]), variational inequalities (J.L. Lions [**2**]). Stochastic Navier–Stokes equations are studied in A. Bensoussan and R. Temam [**1**], C. Foias [**1**], C. Foias and R. Temam [**5, 9**], M.I. Vishik and A.V. Fursikov [**1, 2, 3**]. Optimal control problems for systems governed by the Navier–Stokes equations appear in M. Cuvelier [**1**] (see the end of Appendix III for more recent results).

The difficulties encountered in the mathematical theory of the Navier–Stokes equations lead several authors to reconsider the fluid mechanic hypotheses leading to these equations and to propose new models with a better mathematical behavior; see S. Kaniel [**1**], O.A. Ladyzhenskaya [**1**].

Similar models involving other equations (most often the Navier–Stokes equations coupled with other equations) are: the convection equations whose treatment is almost identical to the treatment of the Navier–Stokes equations, several fluid models, pollution (G. Marshall [**1**]) or blood models (C.S. Peskin [**1**]), and oceanography models (having the appearance of a concentration equation). More elaborated are the magnetohydrodynamic equations and the Bingham equations (see G. Duvaut and J.L. Lions [**1, 2**]) which are an example of non-Newtonian fluids.

The mathematical theory of the Euler equations has not been developed here. For a treatment based on analytical methods, cf. C. Bardos [**1**], T. Kato [**1, 2**], J.L. Lions [**2**], R. Temam [**10, 12**], V.I. Yudovich [**1**].

Some results related to the behavior of the Navier–Stokes equations as $\nu \to 0$ are given in J.L. Lions [**2**], V.I. Yudovich [**1**]. A similar problem for a model equation related to the Burgers equation is completely studied in C.M. Brauner, P. Penel and R. Temam [**1**], P. Penel [**1**]; cf. also C. Bardos, U. Frish, P. Penel and P.L. Sulem in R. Temam [**12**].

Additional comments to the third (revised) edition

We give here some indications on the most recent result on the theory and numerical analysis of the Navier–Stokes equations. These results are mainly oriented in three directions:

(a) *Existence, uniqueness and regularity of solutions*

For the time-dependent Navier–Stokes equations it is known since the work of J. Leray [**1, 2, 3**] and E. Hopf [**1**] that, provided the data are sufficiently smooth, there exists a unique smooth solution to the initial value problem, which is defined on some interval of time $(0, T*)$, and this solution can be extended for subsequent time as a possibly less regular solution (see Chap. 3, Sec. 3 and 4). We do not yet know whether the solutions remain smooth for all time. Following the idea of B. Mandelbrot [**1, 2**], there has been some recent studies on the Hausdorff dimension of the set of singularities of solutions (the set where the velocity is infinite): see V. Scheffer [**1**]–[**4**], C. Foias and R. Temam [**4**] and the most recent article by L. Caffarelli, R. Kohn and L. Nirenberg [**1**] which contains the best available estimates for the Hausdorff dimension of the singular set.

Other recent results on the existence and regularity of solutions include:

- The study of the set of stationary for the flow in a bounded domain (C. Foias and J.C. Saut [**2**], C. Foias and R. Temam [**2, 3**], J.C. Saut and R. Temam [**2**]).
- The existence and the regularity of solutions corresponding to non-smooth data, and in particular a non-smooth domain; this applies to classical situations like the Couette–Taylor flow or the flow in a cavity; see D. Serre [**2, 3**]. Let us mention also for the flow in an unbounded domain the result of D. Serre [**1**] who finds, in some cases, a whole straight line of solutions (in the function space) which is rather unusual for a non-degenerate nonlinear problem.
- Some new a priori estimates for the weak solutions to the time dependent Navier–Stokes equations, implying that the L^∞-norm is L^1 in time ($\boldsymbol{u} \in L^1(0, T; L^\infty(\Omega)^3)$ in dimension of space 3); see C. Foias, C. Guillopé and R. Temam [**1**].
- The derivation of the *compatibility conditions* which are the necessary and sufficient conditions on the data for the regularity of the solution of the time dependent equations near $t = 0$ (of course this has noting to do with the possible singularities at time $t > 0$); see R. Temam [**15**].

(b) *Long time behavior and turbulence*

If the volume forces are independent of time, then time does not appear explicitly in the Navier–Stokes equations and the equations become an autonomous infinite dimensional dynamical system. A question of interest, in relation with the understanding of the turbulence phenomenon is then the behavior for $t \to \infty$ of the solutions of the time dependent Navier–Stokes equations.

The asymptotic analysis of the Navier–Stokes equations has been recently studied: bounds at infinity for the different norms, number of determining modes (or parameters) for the flow, structure and properties of an attractor, etc. ... See A.V. Babin and M.I. Vishik [**1**]–[**4**], P. Constantin, C. Foias, O. Manley and R. Temam [**1**], P. Constantin, C. Foias and R. Temam [**2**], C. Foias and R. Temam [**4, 10**], C. Foias and J.C. Saut [**1**], C. Guillope [**1**], C. Foias, O. Manley, R. Temam and Y. Trève [**1**], E. Lieb [**1**], D. Ruelle [**1**], R. Temam [**16**], O.A. Ladyzhenskaya [**6, 7**], I.M. Vishik [**2**]

(*c*) *Numerical approximation*

Numerous papers, on the numerical approximation of the Navier–Stokes equations have appeared. They contain in particular investigations on the finite element methods, practical aspects of the implementation of finite element methods, application of the penalty method (see Chap. 1, Sec. 6) to fluid flow problems, study of the behavior of the solution of the Galerkin approximation on a large interval of time: see among many references, M. Bercovier [**1**], V. Girault and P.A. Raviart [**1**], R. Glowinski [**1**], F. Thomasset [**2**], T. Kawai [**1**], J.G. Heywood and R. Rannacher [**1**], P. Constantin, C. Foias and R. Temam [**1**] and the bibliographies contained in these references. Monographs developing other aspects of computational fluid dynamics include M. Holt [**4**], D. Gottlieb and S. Orszag [**1**], R. Peyret and T.D. Taylor [**1**] (see also the bibliographies of these references).

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