STOCHASTIC INTEGRALS

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Dedicated to K. ITÔ

PREFACE

This book deals with a special topic in the field of diffusion processes: differential and integral calculus based upon the Brownian motion. Roughly speaking, it is the same as the customary calculus of smooth functions, except that in taking the differential of a smooth function f of the 1-dimensional Brownian path $t \to b(t)$, it is necessary to keep two terms in the power series expansion and to replace $(db)^2$ by dt:

$$df(b) = f(b)\, db + \tfrac{1}{2} f''(b)(db)^2 = f'(b)\, db + \tfrac{1}{2} f''(b)\, dt,$$

or, what is the same,

$$\int_0^t f'(b)\, db = f(b)\Big|_0^t - \tfrac{1}{2}\int_0^t f''(b)\, ds.$$

This kind of calculus exhibits a number of novel features; for example, the appropriate exponential is $e^{b-t/2}$ instead of the customary e^b. The main advantage of this apparatus stems from the fact that any smooth diffusion $t \to \mathfrak{x}(t)$ can be viewed as a nonanticipating functional of the Brownian path in such a way that $\mathfrak{x}$ is a solution of a stochastic differential equation

$$d\mathfrak{x} = e(\mathfrak{x})\, db + f(\mathfrak{x})\, dt$$

with smooth coefficients e and f. This represents a very complicated nonlinear transformation in path space, so it can hardly be called *explicit*. But it is concrete and flexible enough to make it possible to read off many important properties of $\mathfrak{x}$.

Although the book is addressed primarily to mathematicians, it is hoped that people employing probabilistic models in applied problems will find something useful in it too. Chandrasekhar [1], Uhlenbeck–Ornstein [1], and Uhlenbeck–Wang [1] can be consulted for applications to statistical mechanics. A level of mathematical knowledge comparable to Volume 1 of Courant–Hilbert [1] is expected. Yosida [2] would be even better. Also, some knowledge of integration, fields, independence, conditional probabilities and expectations, the Borel–Cantelli lemmas, and the like is necessary; the first half of Itô's notes [9] would be an ideal preparation. Dynkin [3] can be consulted for additional general information; for information about the Brownian motion, Itô–McKean [1] is suggested. Chapter 1 and about one third of Section 4.6 are adapted from Itô–McKean; otherwise there is no overlap. Itô [9] and Skorohod [2] include about half of Chapters 2 and 3, and Section 4.3, but most of the proofs are new. Problems with solutions are placed at the end of most sections. The reader should regard them as an integral part of the text.

I want to thank K. Itô for conversations over a space of ten years. Most of this book has been discussed with him, and it is dedicated to him as a token of gratitude and affection. I must also thank H. Conner, F. A. Grünbaum, G.-C. Rota, I. Singer, D. Strook, S. Varadhan, and the audience of 18.54/MIT/1965, especially P. O'Neil, for information, corrections, and/or helpful comments. The support of the National Science Foundation (NSF/GP/ 4364) for part of 1965 is gratefully acknowledged. Finally, I wish to thank Virginia Early for an excellent typing job.

H. P. McKEAN, JR.

South Landaff, New Hampshire
1968

CONTENTS

LIST OF NOTATIONS

USAGE: *Positive* means >0, while *nonnegative* means $\geqslant 0$; it is the same with *negative* and *nonpositive*. A *field* is understood to be closed under countable unions and intersections of events. The phrase *with probability* 1 is suppressed most of the time. $C^n(M)$ stands for the class of n times continuously differentiable functions f from the (open) manifold M to R^1; *no implication about the boundedness of the function or of its partials is intended.* f is said to be *compact* if it vanishes off a compact part of M.

a	an extra Brownian motion
A	the Lie algebra of G (Section 4.7)
A	a field including the corresponding Brownian field **B** (Section 1.3)
b	a Brownian motion (Section 1.2)
B	an event
B	a Brownian field (Section 1.3)
c	a constant

d	the dimension, a differential (Section 2.6)
D^n	a class of formal trigonometrical sums (Section 4.2)
$D(G)$	the enveloping algebra of G (Section 4.7)
$\mathbf{D}$	a 1-field (Section 4.1), a Lie or enveloping element (Section 4.7)
∂	a partial, the boundary operator
Δ	a Brownian increment $b(k2^{-n}) - b((k-1)2^{-n})$ (Section 2.5), an interval
$\boldsymbol{\Delta}$	a Laplacian, *e.g.*, $\partial^2/\partial x_1{}^2 + \cdots + \partial^2/\partial x_d{}^2$
e	a nonanticipating Brownian functional (Section 2.2), the coefficients of ∂^2 in G (Sections 3.1, 4.1)
$\mathfrak{e}$	an exit or explosion time (Sections 3.3, 4.3)
$E(f)$	the expectation based on $P(B)$ of the function f
f	a function, the coefficients of ∂ in G (Sections 3.1, 4.1)
$\mathfrak{f}$	a local time (Section 3.9)
g	the coefficients of ∂^0 in G (Section 4.1)
G	a group of fractional linear substitutions (Section 4.6), a Lie group (Section 4.7)
$\mathbf{G}$	an elliptic operator (Sections 3.1, 4.1)
$\mathbf{G}^*$	the dual of $\mathbf{G}$ (Section 4.2)
H	a Hermite polynomial (Section 2.7)
i.o.	infinitely often
j	a compact C^∞ function, a patch map (Section 4.1)
J	the Jacobian $\partial x'/\partial x$ (Section 4.1)
lg	logarithm
$\lg_2$	lg(lg)
L^1	the space of functions f with $\|f\|_1 = \int \lvert f \rvert < \infty$
L^2	the space of functions f with $\|f\|_2 = (\int \lvert f \rvert^2)^{1/2} < \infty$
M	a manifold (Section 4.1)
n	an integer
o	an orthogonal transformation (rotation)
$\mathrm{O}(d)$	the orthogonal group
p	an elementary solution of $\partial u/\partial t = \mathbf{G}^* u$ (Sections 3.1, 4.1)
$P(B)$	the probability of the event B, usually Wiener measure (Section 1.2)
$\mathbf{Q}$	an elliptic operator on a torus (Section 4.2)
r	a Bessel process (Section 1.7)
R	a Riemann surface (Section 4.6)
R^d	d-dimensional number space

$R^n \otimes R^m$	the applications of R^m into R^n
SO(d)	the special orthogonal group [det $o = +1$] (Section 4.7)
sp	spur or trace
t	time
$\mathfrak{t}$	a stopping time (Section 1.3), an intrinsic time or clock (Section 2.5)
T	a torus $[0, 2\pi]^d$ (Section 4.2)
u	a solution of $\partial u/\partial t = \mathbf{G}u$
U	a patch of a manifold (Section 4.1)
w	a point of a covering surface (Section 4.6)
$\mathfrak{w}$	a covering Brownian motion (Section 4.6)
x	local coordinates on a patch (Section 4.1)
$\mathfrak{x}$	a stochastic integral (Section 2.6), a diffusion expressed in local coordinates (Section 4.3)
z	a point of a manifold M (Section 4.1)
$\mathfrak{z}$	a martingale (Section 1.4), a diffusion on a manifold (Section 4.3), a complex Brownian motion (Section 4.6)
Z^1	the rational integers 0, ± 1, *etc.*
Z^d	the lattice of integral points of R^d
$\vee$	maximum
$\wedge$	minimum
$\cdot$	the inner product of R^d
$\times$	multiplication, cross product of R^d
$\otimes$	outer product
$*$	transpose
$\lvert\ \rvert$	the norm on R^d, the bound of an application of R^d
$f^{\blacktriangle}$	$(y - x)^{-1}[f(y) - f(x)]$ $(x \neq y)$, $f'(x)$ $(x = y)$ (Section 3.5)
$\lVert f \rVert_1$	$\int \lvert f \rvert$ except in Section 4.2
$\lVert f \rVert_2$	$(\int \lvert f \rvert^2)^{1/2}$ except in Section 4.2
$\lVert f \rVert_\infty$	the upper bound of $\lvert f \rvert$
[ ]	the integral part of
$\cap$	intersection
$\cup$	union
$\subset$	set inclusion
$\in$	point inclusion
$\uparrow$	increases to
$\downarrow$	decreases to
∞	infinity, the compactifying point of a noncompact manifold.

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REFERENCES†

AM	*Ann. Math.*	*MK*	*Mem. Coll. Sci. Kyôto Univ.*
BSMSP	*Berkeley Symp. Math. Statist. and Prob.*	*MZ*	*Math. Z.*
DAN	*Dokl. Akad. Nauk SSSR*	*PAMS*	*Proc. Amer. Math. Soc.*
IJM	*Illinois J. Math.*	*TAMS*	*Trans. Amer. Math. Soc.*
JMP	*J. Math. Phys.*	*TV*	*Teor. Veroyatnost. i Primenen*
MA	*Math. Ann.*	*ZW*	*Z. Wahrscheinlichkeitsth*

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† Skorohod [2] contains a more complete listing of the Russian literature.

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SUBJECT INDEX

ERRATA

P. 13, end of line 6↑: for $\lg_2 \theta^n$ read $\lg_2 \theta^{-n}$.
P. 24, line 2↑ read: (4) $\int_0^t e f db$ (f missing).
P. 31, line 12, just under display 4: for $t(\Delta)$ (Roman t) read $\mathfrak{t}(\Delta)$ (German t).
P. 41, line 9, under display 2: read $\int_0^t f^{-2} ds$ (without the parenthesis).
P. 67, line 9: read $P[t < \mathfrak{e}^f]$ (i.e., reverse inequality).
P. 113, General note: The application of Poincaré's theorem in display 2 is wrong, as kindly pointed out by D. Sullivan, and this spoils the subsequent proof. This was corrected and the result amplified in T. J. Lyons and H. P. McKean, *Winding of the plane Brownian motion*, Adv. Math. **51** (1984), 212–225, and H. P. McKean and D. Sullivan, *Brownian motion and harmonic functions on the class surface of the thrice-punctured sphere*, Adv. Math. **51** (1984), 203–211. The fact is that Poincaré's sum is not infinite but finite and that the covering Brownian motion on the class surface over the punctured plane wanders off to infinity, with the interpretation that the original Brownian motion in the twice-punctured plane, in its winding about 0 and 1, gets inextricably tangled up, not only from the viewpoint of homotopy (that's easy), but from the viewpoint of homology as well.
P. 124, line 8: read $-\frac{1}{2} \sum_{i \le n} \sum_{j \ne i} (\gamma_j - \gamma_i)^{-1} \partial/\partial\gamma_i$.
P. 124, Step 1: Replace x by γ (3 times) and n by d (4 times) in lines 2, 3, and 4. In line 3, read $M_3 = [\gamma : \gamma_1 = \gamma_2 = \gamma_3 < \cdots < \gamma_d]$.
P. 134, line 8↑, Gangolli reference: for 419, read 219.

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