Decompositions of Manifolds

Robert J. Daverman

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PREFACE

This book is about decompositions, or partitions, of manifolds, usually into cell-like sets. (These are the compact sets, similar to the contractable ones, that behave homotopically much like points.) Equivalently, it is about cell-like mappings defined on manifolds. Originating with work of R. L. Moore in the 1920s, this topic was renewed by results of R. H. Bing in the 1950s. As an unmistakable sign of its importance, the subject has proved indispensable to the recent characterization of higher-dimensional manifolds in terms of elementary topological properties, based upon the work of R. D. Edwards and F. Quinn.

Decomposition theory is one component of geometric topology, a heading that encompasses many topics, such as PL or differential topology, manifold structure theory, embedding theory, knot theory, shape theory, even parts of dimension theory. While most of the others have been studied systematically, decomposition theory has not. Filling that gap is the overriding goal. The need is startlingly acute because a detailed proof of its fundamental result, the cell-like approximation theorem, has not been published heretofore.

Placing the subject in proper context within geometric topology is a secondary goal. Its interrelationships with the other portions of the discipline nourish its vitality. Demonstrating those interrelationships is a significant factor among the intentions. On one hand, material from other topics occasionally will be developed for use here when it enhances the central purpose; on the other hand, applications of decomposition theory to the others will be developed as frequently as possible. Nevertheless, this book does not attempt to organize all of geometric topology, just the decomposition-theoretic aspects, in coherent, linear fashion.

Uppermost in my thinking, from the earliest stages of the book's conception, has been the belief it should be put together as a text, with as few prerequisites as possible, and so it has evolved. Not intended for experts, it aims to help students interested in geometric topology bridge the gap between entry-level graduate courses and research at the frontier. Along the way it touches on many issues embraced by decomposition theory but makes no attempt to be encyclopedic. It depicts foundational material in fine detail, and as more of the canvas is unveiled, it employs a coarser brush. In particular, after the proof of the climactic result, the cell-like approximation theorem, it tends to present merely the cruder features of later topics, to expose items deserving further individual pursuit. All in all, it should equip mature readers with a broad, substantial background for successfully doing research in this area.

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About this book

Decomposition theory studies decompositions, or partitions, of manifolds into simple pieces, usually cell-like sets. Since its inception in 1929, the subject has become an important tool in geometric topology. The main goal of the book is to help students interested in geometric topology to bridge the gap between entry-level graduate courses and research at the frontier as well as to demonstrate interrelations of decomposition theory with other parts of geometric topology. With numerous exercises and problems, many of them quite challenging, the book continues to be strongly recommended to everyone who is interested in this subject. The book also contains an extensive bibliography and a useful index of key words, so it can also serve as a reference to a specialist.