Matching Theory

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Appendix: Developments in matching theory since this book was first published

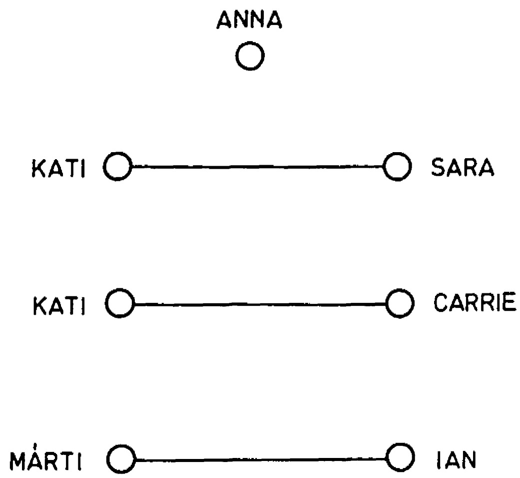

FIGURE P.1.

Preface

Suppose a pharmaceutical firm wishes to test n antibiotics on n volunteer subjects. However, preliminary screening has shown that certain subjects are allergic to certain of the drugs. Can an experiment be designed in which each subject takes exactly one of the antibiotics to which he/she is not allergic and each drug is taken by exactly one subject?

Let us model this situation using a bipartite graph G in which the two point classes consist of the n subjects and the n antibiotics respectively and let us agree to join subject to drug if and only if the subject is not allergic to the drug. Then the answer to the question posed is "yes" if and only if graph G has a **perfect matching** (or **1-factor**), that is, a pairing of subjects to drugs which uses each subject and each drug once and only once.

Next suppose one has two computers available and p jobs to be processed on these machines. We will assume that any job can be run on either machine. Indeed, we may assume that the computers are identical. Let us also suppose that the p jobs are partially ordered in the sense that for any two (different) jobs J_i and J_k, $J_i \leq J_k$ if J_i must be completed before J_k can be started (by either computer). If all jobs require an equal amount of time to complete, what is the shortest possible time sufficient to run all p jobs?

Let us model this situation using an undirected graph G as follows. Let the points of G be the jobs $J_1, \ldots, J_p$ and let us join J_i and J_k with a line if and only if they are incomparable in the partial order. (In other words, if they can be run simultaneously.) Now it is clear that to design an optimum schedule, we must use both machines simultaneously as often as possible; that is , we must find a matching of largest cardinality in G. (See Fujii, Kasami and Ninomiya (1969, 1971) and Coffman and Graham (1972).) This problem belongs to the class of so-called **maximum cardinality matching** problems. Note also that in this case, unlike the drug testing example above, the graph modeling the problem is no longer bipartite.

Now let us get our hands dirty! Suppose we are drilling for oil. Seismic (and/or other) information has been supplied to us which gives accurate locations of oil deposits. The existing technology at the time of this writing is such that one can drill until one deposit of oil is tapped and then drilling may be continued in the same hole to a second deposit situated at a location somewhat deeper, but not necessarily directly beneath, the first deposit. (See Figure P.2.). One can then bring up oil from both deposits through two concentrically placed pipes in the same drill hole. (For further technical comments see Devine (1973).)

One can associate a number with each pair of deposits representing the cost of tapping them both in one drilling operation. Practical impossibility of pairing can be reflected by assigning a very large (or even infinite) value to a pair. For the sake of simplicity, let us assume that an even number of deposits is under consideration so that all deposits may be simultaneously paired off.

The task is then to find a set of pairs (that is, a "perfect matching") of deposits in which all deposits are paired and so that the sum of the costs is minimum. In other words, we seek a **minimum weight perfect matching** in the associated complete graph.

To put our next example in a bit more facetious form, suppose that $2n$ students arrive at Nashpest University at the beginning of the school year and are to be assigned to n 2-person dormitory rooms. The housing office has given each student a roster of all $2n$ students and asked each student to check off those other students on the list with whom they are acquainted. This information is then fed into the University computer. But, alas, on registration day the computer "crashes"! (Such has been known to happen on occasion!) The Dean of Housing, in despair, randomly assigns two students to each room. What is the probability that only students acquainted with each other are paired by the Dean's

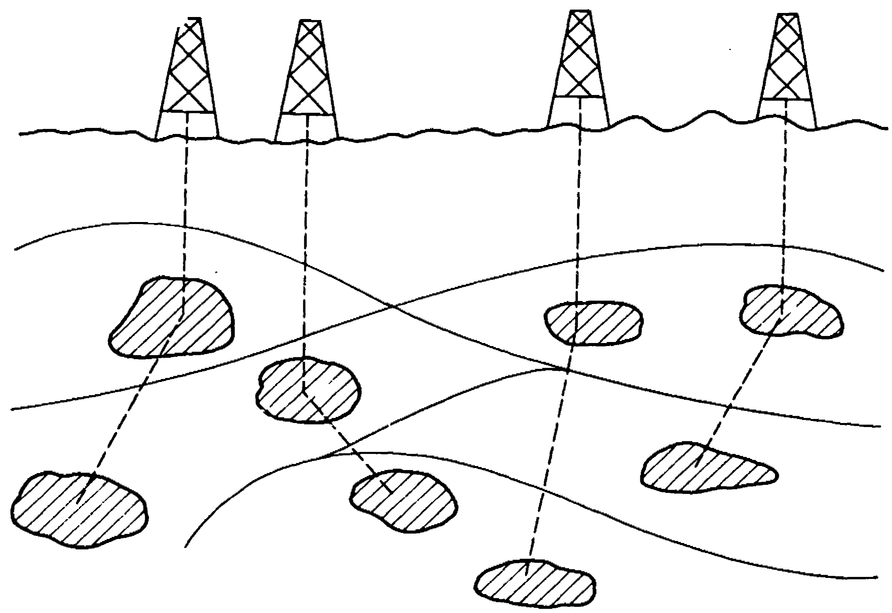

FIGURE P.2.

random assignment?

Let us build a graph G with the $2n$ students as points and agree to join two students if they are acquainted with each other. The answer to the room assignment question above is then clearly $\Phi(G)/(2n-1)!!$ where $\Phi(G)$ denotes the number of perfect matchings in G.

Sometimes problems which are not directly convertible into matching problems can nevertheless be solved with the help of matching theory. **The Chinese Postman Problem** (see Meigu Guan (1962)) was first presented as follows. A postman must deliver mail along all streets of a town. How can he leave the post office, perform the "swift completion of his appointed rounds" and return to the post office having traversed a minimum distance?

Here there is no longer an obvious translation of the problem directly into a matching problem. But we shall see later that this problem can be efficiently solved by first solving a set of shortest path problems and then solving a certain minimum weight perfect matching problem.

Let us now modify the last problem a bit. Suppose our postman's task is to pick up mail from boxes located at street intersections only.

Again, he wants to find a shortest route. The main difference between this problem and the previous one is that now he has to visit all *intersections*, but does not have to traverse all the *streets*. In a more common formulation of this problem, a travelling salesperson has to visit all capitals of states in the U.S. (or all the counties of Hungary) in such a way as to minimize the length (or time, or cost) of the route. This problem is known as the **Travelling Salesman Problem** and it has received quite a lot of attention in the "real" world.

As a special case, the salesperson may only be interested in finding a tour which avoids certain connections. In such a case we retain the lines representing allowable connections and delete all other lines. The problem then reduces to finding a Hamilton cycle (that is, a cycle through all the points) in the given graph. This problem is called **The Hamilton Cycle Problem.**

Superficially, the Travelling Salesman Problem seems quite similar to the Chinese Postman Problem and to the Weighted Perfect Matching Problem. However, the Travelling Salesman Problem (and even its special case, the Hamilton Cycle Problem) turn out to be significantly more difficult. More particularly, the Chinese Postman Problem is solvable in polynomial time, but the Travelling Salesman Problem is "NP-hard" (i.e., it belongs among the hardest combinatorial problems). So here we have an example of a generalization of matching to a truly more difficult problem.

But matching theory is often used as a stepping stone to the Travelling Salesman Problem. A "tour" for a salesperson is a connected spanning subgraph in which all points have degree 2. If we drop the connectedness assumption we arrive at the concept of a **2-factor**. In this way the 2-factor is a natural extension of the notion of a perfect matching and in fact the 2-factor problem can be reduced to finding a perfect matching. In this light the Matching Problem can be viewed as a "relaxation" of the Travelling Salesman Problem.

As our last example, let us describe an important and well-known combinatorial problem, but one not usually thought of as being related to matching. A telephone company wishes to provide service to each of n cities so that any city may call any other. How can the cities be connected so that construction costs are minimized?

Here the solution is clearly a minimum weight spanning tree. This problem can be rather easily solved by the so-called Greedy Algorithm (Borůvka (1926a, 1926b), Kruskal (1956)) in which at each step one selects the cheapest "allowable" line.

But this "greedy" procedure does not work for the minimum weight perfect matching problem and in fact matching problems are nearly always more difficult to solve. Why? In modern parlance the reason lies in the fact that the spanning trees of a graph form a "matroid", whereas the perfect matchings do not. Nevertheless we shall see that matroids do indeed arise at many points in the study of matchings.

So we see that matching and some of its close relatives model rather a lot of applied problems. But does this warrant writing a book on the subject? In a sense, matching problems seem somehow to be of the "proper" level of difficulty. On the one hand, most of them are solvable problems, but to solve them certainly requires non-trivial methods. Indeed matching theory has played a catalytic role over the past hundred years or so in developing a number of new and more general combinatorial methods. To see more concrete manifestations of this, let us take a look at some history.

It is fascinating (if not downright dangerous!) to probe the roots of any family tree. Although it can be argued that such famous names as Euler, Kirchhoff and Tait can be found in the historical shadows of matching theory, we shall take as the two principal "founders" of the discipline the Dane, Julius Petersen and the Hungarian, Dénes König. Although their interests certainly overlapped, it is perhaps helpful to identify Petersen with the earliest study of *regular* graphs (that is, graphs having the same degree at each point) and König with *bipartite* graphs.

In an 1891 paper, Petersen considered an algebraic factorization problem due to Hilbert (1889) and reformulated it as a factorization problem for graphs. He set as his general problem the task of deciding which regular graphs have a non-trivial factorization into smaller regular spanning subgraphs the union of which is the parent graph.

He then proceeded to prove that any graph regular of even degree can be expressed as the union of line-disjoint 2-factors. This result is intimately related to the famous result of Euler who showed in his celebrated paper on the Königsberg Bridge Problem (1736) that one can traverse all lines of a graph once and only once and return to the starting point without stopping if and only if the graph under consideration is connected and has all degrees even. Petersen does not mention Euler in his paper and indeed we do not know if he was aware of the Euler result, already at that time more than 150 years old.

Petersen quite accurately observed that factorization of graphs regular of *odd* degree is a more difficult problem. He then proved that any connected 3-regular graph having no more than two cutlines has a perfect

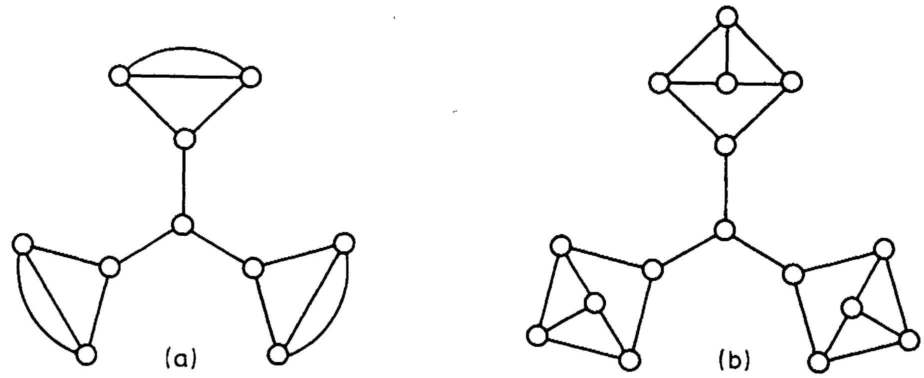

FIGURE P.3. Sylvester's graphs

matching (and thus is decomposable into a 1-factor and a 2-factor). Petersen pointed out that two cutlines was best possible in the sense that there are 3-regular graphs with three cutlines and no perfect matching. He gave the example in Figure P.3(a) and attributed it to Sylvester. (Figure P.3(b) contains the corresponding graph without multiple lines.)

Petersen's proof of his decomposition theorem for 3-regular graphs was successively simplified by Brahana (1917-18), Errera (1922) and by Frink (1925-26). The last of these contained a slight error which was corrected in König's book (1936). Petersen's work was extended to other regular graphs by Bäbler (1938, 1952, 1954), Gallai (1950) and by Belck (1950). This line of research culminated in Tutte's work which we shall discuss later.

Petersen also became interested in the famous Four Color Conjecture which had been formulated about fifty years before. We make no attempt in our book to chronicle the work on this problem, but refer the reader to Biggs, Lloyd and Wilson (1976) for a lively account of the early years of this conjecture. The conjecture says that the regions of any planar map (that is, the faces of any plane graph) can be colored in four colors in such a way that no two regions sharing a common boundary line receive the same color.

To connect this problem to Petersen we must introduce the Scot, P.G. Tait to our story. It already had been observed by Cayley and Kempe that in order to prove the Four Color Conjecture true in general, it would be enough to prove it true for 3-regular (that is, "cubic") planar graphs. In a series of three papers (1878-1880a, 1878-1880b, 1880) Tait then took up the problem (in a rather foggy manner). He made the important observation (1878-1880b) that 4-coloring the regions of a cubic planar map was equivalent to 3-coloring the *lines* of the map and then he turned his attention to factoring cubic planar graphs.

Tait provided an example to show that if a cubic graph has a cutline it may not be factorizable into three perfect matchings. But he then claimed that if the cubic graph were "polyhedral" such a factorization could always be carried out. (Steinitz (1922) later showed that "polyhedral" means precisely 3-connected and planar.) He then outlined various approaches to prove his claim which, of course, did nothing of the kind.

Tait also made a second conjecture, namely that every cubic polyhedral graph contains a Hamilton cycle. He then pointed out, accurately enough, that if this conjecture were true, 3-line colorability of all cubic polyhedral graphs would follow immediately for one could color the lines of the (necessarily even) Hamilton cycle with two colors and all other lines with the third color. This second conjecture of Tait enjoyed a long life too until Tutte (1946) found the first counterexample.

Of course had Tait's second conjecture been true, the Four Color Conjecture would have been proved. But as all mathematicians know, this celebrated problem remained open for nearly 100 years more until it was finally settled in the affirmative by Appel and Haken (1977) and Appel, Haken and Koch (1977).

But let us now return to Petersen who in 1898 published a second paper in which he responded to Tait's remarks on cubic graphs. The most significant contribution made to graph theory in this paper was undoubtedly the introduction of a non-planar graph which was cubic and had no cutlines, but which could not be decomposed into three disjoint perfect matchings. This graph, nowadays deservedly called the "Petersen graph", is perhaps the single most famous graph in existence (or the most notorious, depending upon what it has done to the reader's conjectures!) The graph is shown in Figure P.4. The reader will encounter this graph several times in the present book as well.

In the meantime, however, a second stream of historically important results relevant to matching had begun to flow. The emphasis this time was on *bipartite* graphs. The German algebraist, Frobenius, who had become interested in reducibility properties of determinants, proved the following result (1912). (In order to keep things sorted out subsequently, let us call this Theorem F-1.) Consider an $n \times n$ matrix M such that every entry is either zero or a variable, all variable entries being different from each other. Then the determinant of the matrix is a reducible (that is, factorizable) polynomial of these variables if and only if there is an

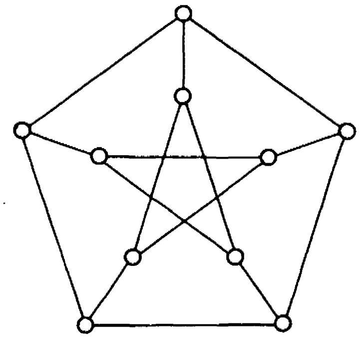

FIGURE P.4. The Petersen graph

integer $p, 0 \le p \le n$, and a permutation of the rows and columns of M which results in a block of zero entries of size $p \times (n-p)$. Frobenius' proof of F-1 was more complicated than need be and König (1915), having cast the problem in terms of bipartite graphs, gave a shorter proof.

In retrospect, König's translation of the above problem into graph-theoretical terms was indeed quite straightforward, but was to prove to be of considerable significance. Let the rows of a square matrix M be $r_1, \ldots, r_n$ and the columns be $c_1, \ldots, c_n$. Form a bipartite graph by joining r_i to c_j if and only if entry m_{ij} of matrix M is not zero. It is now immediate that there is a one-to-one correspondence between non-zero expansion terms in the determinant of M and perfect matchings in the bipartite graph constructed from M as above.

We should remark parenthetically here that relationships between graphs and linear algebra go much deeper than the matrix-bigraph model just mentioned. We will treat a number of these relationships in this book.

In the following year in two nearly identical papers — one in German (1916a), the other in Hungarian (1916b) — König proved that every doubly stochastic matrix with non-negative entries must have a non-zero term in its determinant. (An $n \times n$ matrix is **doubly stochastic** if the $2n$ row and column sums all have the same value.) The proof given is for integer matrices, but König pointed out that the result is clearly extendable to the rationals and then to the reals. In the proof he gave for integers he considered once again the corresponding bipartite graph and observed that the matrix is doubly stochastic if and only if this bigraph is regular. Also in his twin 1916 papers, König proved that every bipartite graph which is regular of degree k is the union of k disjoint perfect matchings. It is a trivial observation — once these results are proved — that any doubly stochastic matrix with non-negative elements must be a convex sum of permutation matrices. Couched in these terms, the result was to be independently rediscovered some forty years later by Birkhoff (1946) and von Neumann (1953) and has come to be known as the Birkhoff–von Neumann Theorem.

It is in the 1916 papers that one finds the first proof of yet another famous theorem for bipartite graphs. We shall call it the König Line Coloring Theorem. (See Theorem 1.4.18 of the present book.) This result says that in any (not necessarily regular) bipartite graph the chromatic index is equal to the maximum degree of all the points.

At this point the plot begins to thicken a bit! In 1917, Frobenius published his own simpler proof of Theorem F-1. He obtained his proof using the following lemma which we shall denote by F-2. Consider an $n \times n$ matrix M in which every entry is either zero or a variable, different entries being different variables. Suppose that the determinant of M vanishes identically as a polynomial of its non-zero entries. Then there is an integer $p, 0 < p < n$, and permutations of the rows and columns of M such that the resulting matrix contains a block of zeros of size $p \times (n-p+1)$.

It is somewhat surprising that the question as to whether a determinant is *irreducible* was studied much earlier by Frobenius than the more natural question as to whether it is *identically zero.* We all learned in school that a polynomial is identically zero if and only if upon expanding it, all terms cancel. From today's more algorithmic point of view, this is not a satisfactory test since it may involve exponentially long computations. But from the classical point of view, irreducibility was a much more intriguing question.

Frobenius did not cite König's 1915 proof of Theorem F-1, even though König contended that he had sent his proof to Frobenius in German translation. (See footnote 2, page 240 of König's book (1936).) However, Frobenius did cite König's paper (1916a), but he then dismissed König's graph-theoretic formulation of these determinant questions as being of little value, thereby heaping a few more coals upon the fire! For König's ripost the interested reader is referred to the above-mentioned footnote or to König (1933). For further commentary on this issue, see Schneider (1977).

Lemma F-2, in addition to its role in obtaining the short proof of F-1 referred to above, is of considerable interest in its own right, for if one translates it into the language of bigraphs, it gives a necessary and sufficient condition for a bipartite graph to have a perfect matching. In this guise it has come to be called the Marriage Theorem.

Suppose we have n men and n women and we wish to arrange n marriages (without bigamy, polyandry or homosexual relationships). Let us suppose further that we wish to marry only men and women who are acquainted with each other. The Marriage Theorem states that this is possible if and only if for each $k, 1 \leq k \leq n$, each set of k men collectively knows at least k women.

This theorem was the forerunner of one of the best known results in bipartite matching, the Theorem on Distinct Representatives due to Philip Hall (1935). Hall's Theorem was first stated in terms of sets and may be simply expressed as follows. Let $S_1, \ldots, S_n$ be a finite collection of sets. Then there is a set of distinct elements $x_1, \ldots, x_n$ such that $x_i \in S_i$ if and only if for each $k, 1 \leq k \leq n$, the union of any k of the S_i's contains at least k elements. It is clear that the Marriage Theorem of Frobenius follows from this result immediately. (In fact, the two are equivalent, but more about that in Chapter 1.)

At this point we must backtrack about four years to 1931 when the first proof appeared of the so-called König Minimax Theorem for bipartite graphs (Theorem 1.1.1 of our book). (See König (1931) and (1933) for Hungarian and German language versions respectively.) This result says that in a bipartite graph G the size of a largest matching is equal to the size of a smallest set of points which together touch every line of G. In the same year, Egerváry (1931) generalized this result to graphs with non-negative weights on each line. Such "minimax" results as these (the classical theorem of Menger (1927) on graph connectivity and the Max-Flow Min-Cut Theorem of Ford and Fulkerson (1956) and Elias, Feinstein and Shannon (1956) on network flows are but two of many others) are of great significance in our opinion and we shall emphasize this type of result throughout the book. Indeed the importance of such results grows daily in various branches of combinatorics, due largely to increasing use of linear programming to formulate and solve many combinatorial problems. (See Schrijver (1983a) for an excellent up-to-date survey of minimax results in combinatorics.)

In the first textbook on graph theory ever written, König (1936) showed that Menger's theorem on connectivity, the Marriage Theorem of Frobenius and P. Hall's Theorem on Distinct Representatives all follow from his minimax theorem. In fact, we shall see in Chapter 1 that these four results, together with the Max-Flow Min-Cut Theorem and an important theorem on partially ordered sets due to Dilworth (1950), are all *equivalent*.

For further interesting historical comments on the development of matching theory prior to the appearance of König's book in 1936, the reader is referred to the book of Biggs, Lloyd and Wilson (1976). Also Gallai has written a very interesting account of the life and work of König (1964b, 1978).

Very few papers on matching theory appeared during the years of the Second World War. In 1942, however, Rado published a paper in which he generalized P. Hall's theorem to independent systems of distinct representatives in Euclidean vector spaces and thereby established the first link between matchings and *matroids.* In 1945, Marshall Hall applied the notion of systems of distinct representatives to extend Latin rectangles to Latin squares.

In the immediate postwar period several exciting results burst upon the scene and from these matching theory received a vigorous boost. In 1947, Tutte proved a theorem characterizing those general (that is, non-bipartite) graphs with perfect matchings. (See Theorem 3.1.1 of this book.) This elegant result which, with the advantage of hindsight, is the "natural" generalization of the bipartite Marriage Theorem has become the cornerstone of matching theory in the non-bipartite case. Suppose a connected graph G contains a set S of points such that $G - S$ has more than $|S|$ components having an odd number of points each. It is then clear that G cannot contain a perfect matching. The crucial contribution of Tutte's Theorem was to prove the converse true; that is, if G has no separating set S with this property, then G must have a perfect matching. This theorem was destined to become an archetypal example of a "good characterization" in the language of algorithmic complexity theory. The latter discipline was born only in the 1970's as mathematicians scrambled to keep up with the explosive entry of the computer upon the scientific scene along with the accompanying surge of interest in algorithms. But before considering computers and algorithms per se, let us mention a few more important non-algorithmic results forthcoming during the two decades immediately following World War II.

In the early 1950's, Tutte proved his so-called "f-factor theorem" which has its roots back in the 1890's in the earlier work of Petersen. The concept of an f-factor and the closely related "degree-constrained subgraph problem" were also studied by Ore and Gallai in the 1950's and early 1960's. In 1955, Ore published his "defect" version of P. Hall's Theorem for bipartite graphs and in 1958, Berge obtained the analogous "defect" version of Tutte's perfect matching theorem for the non-bipartite case.

The first two textbooks in graph theory after König were due to Berge and Ore respectively. Berge's book appeared in its French edition in 1958 and in an English translation in 1962, the same year that Ore's book in English was published. Together they introduced graph theory to a much wider audience (including the second author of the present book) and their appearance set the stage for the incredible growth of graph theory in both breadth and depth seen in the last two decades.

In the late 1950's and early '60's, Dulmage and Mendelsohn published a series of papers in which they worked out a canonical decomposition theory for bipartite graphs in terms of maximum matchings and minimum point covers. Their work was motivated by questions concerning matrices.

The year 1964 marked the appearance of a paper by Gallai which contained one of the central results of this book. (See Chapter 3.) In this paper, Gallai established the existence of a *canonical* decomposition theory of any graph in terms of its maximum matchings. An efficient method to obtain this decomposition was provided by the polynomial matching algorithm for general graphs due to Edmonds which appeared in 1965. Hence we have chosen to call this important result the Gallai-Edmonds Structure Theorem.

One of several important degenerate cases for the Gallai-Edmonds theorem arises when the graph in question has a *perfect* matching. However, Kotzig had already begun to lay the foundations for a canonical treatment of these graphs in a series of papers which appeared in 1959 and 1960. It is unfortunate that these important papers remained more or less unnoticed since they were written in Slovak. In these publications, Kotzig introduced a certain binary relation on the point set of any graph having a perfect matching. For an important special class of such graphs, the so-called "elementary" graphs, this relation is an equivalence relation and thus induces a canonical partition of the point set.

But let us now return once more to the period immediately following World War II and take a second historical tack. Computers immediately focused attention on the development of algorithms naturally enough, but if we turn our attention back to matching theory, we see that a fundamental algorithmic question has been with us since the earliest days of the subject. Its importance is perhaps belied by its simplicity of statement: how do you find a perfect (or maximum) matching?

It is no surprise that the first matching algorithm did not spring forth "fully coded" from any one forehead! The rudiments for finding a maximum matching in a bipartite graph had already appeared in the works of König and Egerváry in the 1930's. Kuhn (1955) and M. Hall (1956) presented the first formal procedures for finding a perfect matching in a bigraph. It seems to have been Kuhn who at this time first used the phrase "Hungarian Method" to distinguish algorithms of this type.

At almost the same time, Ford and Fulkerson published the first papers on the theory of network flows (1956, 1957, 1962). Flow theory immediately became a substantial new tool in combinatorial applications of all kinds. Flows can be easily visualized and for our purposes flow theory is of great importance because it can be used to prove most results in bipartite matching.

Matching in non-bipartite graphs turned out to be substantially more difficult and almost another decade passed before Edmonds (1965a) found the first efficient algorithm to find a maximum matching in such a graph. This algorithm also motivated Edmonds to propose *polynomial time* as a measure of "goodness" of algorithms, a point of view which has proved extremely fruitful in theoretical computer science.

At this point we must pursue yet another historical branch. Another product of the early post-war years which has had enormous impact upon not only mathematics itself, but upon nearly every quantitative area of science, is *linear programming.*

In the late 1930's, Kantorovich seems to have been the first to cast linear programming as a mathematical theory in its own right, but his work remained unnoticed in the West. Moreover, no complete algorithm for solving a linear program had yet appeared.

Motivated by World War II planning activities, Dantzig and von Neumann independently discovered and developed the new subject. The important concept of *duality* was introduced by von Neumann (1947). Dantzig (1951) gave the infant discipline a giant practical boost when he introduced the algorithm known as the Simplex Method. This method has solved nearly all real-life linear programming problems far more efficiently than any other method known before or since. For a more extensive historical review of linear programming, see Dantzig's book (1963) and his more recent historical article (1983).

A link between linear programming and matching theory was soon discovered. In 1955, Kuhn published the first of several papers (1955, 1956) in which he cast *bipartite* matching — weighted and unweighted — in the primal-dual setting of linear programming for the first time. To obtain a combinatorial minimax theorem from linear programming duality, one needs a sufficient condition for the integrality of the optimum solutions. If the graph is bipartite integrality follows easily. The first more general sufficient condition for integrality to be discovered — *total unimodularity* — was found by Hoffman and Kruskal (1956). In 1958, Gallai used unimodularity and the duality theorem of linear programming to derive a number of minimax results including the Menger, Dilworth and Egerváry theorems as well as Max-Flow Min-Cut. Hoffman (1960) also accurately predicted that linear programming would become an important general tool in handling combinatorial optimization problems.

The extension of the linear programming approach to the case of *non*-bipartite graph matching turned out to be quite difficult, however, and it required the introduction of a new technique, namely the technique of describing the convex hull of incidence vectors of matchings by linear inequalities. Such a set of linear inequalities was found by Edmonds (1965b). His result allows us to obtain various minimax theorems in matching theory as special cases of the Duality Theorem of linear programming. This approach initiated the study of other combinatorially defined polyhedra and has led to a whole new branch of combinatorial mathematics — polyhedral combinatorics. (For a very recent survey of this discipline see Pulleyblank (1983).)

It is also a natural idea to try to combine this result with some linear programming algorithm to obtain a maximum matching algorithm. This is not straightforward because of the large number of inequalities involved. However, a recently discovered new method for solving linear programs, the so-called *Ellipsoid Method* (Šor (1970, 1977), Judin and Nemirovskiĭ(1976), Hačijan (1979)) can be used to turn a polyhedral description of the convex hull of matchings into a polynomial-time algorithm for maximum matching. (For more details see Grötschel, Lovász and Schrijver (1981).)

Besides the existence and optimization problems mentioned above, there are other important aspects of matching theory. The study of the Ising model for ferromagnetic materials led Kasteleyn (1961, 1963) to the problem of *enumerating perfect matchings* in graphs and he was able to solve this problem for planar graphs. Later, L. Valiant (1979a) proved that the enumeration problem for perfect matchings is NP-hard in general, but useful upper and lower bounds have been obtained. Recent work of Heilmann and Lieb (1970, 1972) and Godsil (1981b) on the generating function for the number of matchings of various sizes relates this problem to determinants and thereby recalls the work of Frobenius dating back to the beginning of this century.

Our historical sketch now brings us to the present book. The first chapter deals with bipartite matching, as was the case historically. In particular, we present first the König Minimax Theorem and then the Marriage Theorem of Frobenius and the result on distinct representatives due to P. Hall. The bipartite matching algorithm know as the Hungarian Method is presented next. For treating bigraphs not having perfect matchings, the concepts of deficiency and surplus are then introduced. The fact that they are supermodular and submodular functions respectively, is discussed and this leads us to consider such functions in the more general framework of matroids. We close the chapter with some of the many consequences of the König–Hall–Frobenius results; König's Line Coloring Theorem and Dilworth's Theorem being only two examples.

In Chapter 2 we develop enough from the theory of network flows to show that most bipartite matching results can be expressed and solved within this framework. In addition the idea of a flow-equivalent tree which is needed later is introduced in this chapter.

Chapter 3 contains fundamental results for the non-bipartite case such as Tutte's Theorem on perfect matchings and Berge's "defect" version thereof. Next we develop the Gallai-Edmonds Structure Theory. The Gallai-Edmonds theory helps us reduce the study of the structure of maximum matchings to three disjoint classes of more special graphs: factor-critical graphs, positive surplus bipartite graphs and graphs having perfect matchings. The last of these three families is further reduced to the study of those which are "elementary". A graph with a perfect matching is **elementary** if the union of all its perfect matchings forms a connected subgraph.

In Chapter 4 we study elementary bipartite graphs and in Chapter 5 we undertake the investigation of elementary graphs in general. We produce a further decomposition of the latter class into smaller elementary *bipartite* graphs and into a new type of elementary graph called "bicritical". This brings us to a frontier, so to speak, as far as decompositions which are "canonical" are concerned. Factor-critical and bicritical graphs are taken up next and quite a lot of useful information about both families is gained from the so-called "ear decomposition" results presented. But such ear decompositions, although quite useful, are not *canonical.* Canonical theories for the decomposition of factor-critical and bicritical graphs do not yet exist.

In Chapter 6 we generalize the idea of matchings to subgraphs having all degrees at most two — the so-called **2-matchings**. We see that the corresponding generalization of König's Minimax Theorem to 2-matchings holds for *all* graphs and not just for those which are bipartite! The 2-matching analogues of elementary and bicritical graphs are introduced.

In many ways these 2-matchings are easier to handle than ordinary (1-) matchings; so it is quite reasonable to ask, for example, how maximum 2-matchings might be used to obtain maximum 1-matchings. The answer is not yet known. In another direction of generalization, we show how 2-matchings can be used to give good characterizations of non-bipartite graphs which satisfy the König minimax equation.

Next we discuss the Chinese Postman Problem from the point of view of 2-matchings and finally, we close out the chapter with a collection of other problems which are reducible to matching problems of one kind or another.

König's Minimax Theorem is a special integer-valued instance of the more general Duality Theorem of linear programming. In Chapter 7 we formulate more general matching problems as linear programs. We present Edmonds' approach to determining the facets of the associated convex polytope spanned by the binary incidence vectors of *all* matchings in a graph. We shall study this so-called *matching polytope* $M(G)$ as well as an assortment of related polyhedra: the *fractional matching polytope*, $FM(G)$, the *vertex packing polyhedron* $VP(G)$ and equivalently, the *point cover polyhedron* $PC(G)$, the *fractional point cover polyhedron* $FPC(G)$ and, finally, the *perfect matching polytope* $PM(G)$. Our knowledge of these varies from considerable for the matching polytope $M(G)$ to very little in the case of $VP(G)$. We shall return to $VP(G)$ in Chapter 12.

König's initial investigations of perfect matchings were motivated by related questions for determinants and in Chapter 8 we return to this relationship and extensions thereof. Because of algebraic sign problems with the expansion terms, the determinant cannot be used to enumerate perfect matchings in a bigraph in general. This difficulty can be overcome, in a sense, by switching to the *permanent* function, but then new problems replace the old. The permanent is notoriously hard to handle! Nevertheless, we study bounds of various types for the permanent function and use them to obtain bounds for the number of perfect matchings in a regular bigraph.

We extend our considerations to the non-bipartite case by introducing a third matrix function–the *Pfaffian*. Some probabilistic considerations are then discussed before moving on to a discussion of the *matching polynomial*. This polynomial is related to the better-known "characteristic polynomial" of a graph, and in fact for some graphs — trees for example — the two coincide. The matching polynomial is then applied to two topics from theoretical chemistry and physics, so-called *topological resonance energy*, as well as to the *Ising model* for magnetic materials.

Whereas the bounds on the permanent developed earlier apply only to graphs which are regular, we close this chapter by applying results from earlier chapters to obtain new lower bounds for graphs which are not necessarily regular.

In Chapter 9 we present and analyze the *Edmonds Matching Algorithm.* Three other algorithmic approaches are also discussed. Two of these turn out to be polynomial, namely a routine based upon the Gallai-Edmonds Decomposition and a second using the very recently developed Ellipsoid Method for linear programming. The third routine, due to Padberg and Rao (1982) is not polynomial in the worst case, but it seems to be competitive with the algorithm of Edmonds in practice.

In Chapter 10 we generalize the degree one restriction of perfect matchings by investigating spanning subgraphs having a prescribed degree $f(v)$ at each point v. We call these *f-factors.* We begin by showing that in fact one can reduce the "f-factor problem" to the "1-factor problem", that is, to the perfect matching problem. Then we proceed to obtain results for f-matchings analogous to the Gallai-Edmonds decomposition results for 1-matchings.

A classical question in graph theory asks: when is a given sequence of non-negative integers realizable as the sequence of degrees of a graph? This may be viewed as a special case of the f-factor problem where the graph in question is complete. We address questions of this type as we bring Chapter 10 to a close.

In Chapter 11 we discuss some extensions of (non-bipartite) matching theory to the more general setting of matroids. (The reader will have seen by the time he reaches Chapter 11 that we have been only partially successful at keeping matroid theory out of the first ten chapters!) We formulate a number of matroid problems which turn out to be equivalent and which are collectively called the *Matroid Matching Problem.* Examples drawn from the disciplines of architecture and electrical engineering show that this extension of matching theory has important applications.

In our final chapter we take up the study of *vertex packing,* that is, the study of independent sets of *points.* Since matchings in any graph G correspond to independent sets of points in the associated line graph $L(G)$, one can take the point of view that all matching problems are just problems about vertex packing in the special subclass of all graphs — the line graphs. Unfortunately this does not help much! Our hopes are further dashed upon learning that the vertex packing problem is NP-complete, whereas matching, as we know, is polynomial.

But progress has been made on vertex packing and a number of interesting results have been obtained, mainly in the study of so-called *τ-critical graphs.* We present most of what is currently known about these interesting, but difficult, graphs including a finite basis theorem.

We next revisit the vertex packing polytope. Facet determination, as was done for the matching polytope in Chapter 7, seems hopeless here, but for bipartite graphs and for the more general class of "perfect" graphs this polytope has a nice description which we present.

It turns out that perfect graphs are also closely related to another natural generalization of matching, namely *hypergraph matching.* A hypergraph is a generalization of the idea of a graph in which a line may have more than two endpoints. The matching problem can be generalized to hypergraphs in a natural way: find the maximum number of disjoint "lines" in a hypergraph. While this problem is NP-complete, various generalizations of König's Minimax Theorem to hypergraphs do exist. We discuss briefly this problem of hypergraph matching.

One interesting class of graphs which contains the line graphs and for which vertex packing is polynomially solvable is the class of *claw-free* graphs. We conclude the chapter — and the book — by presenting an algorithm for this problem.

Now we must say a few words about the "Boxes" inserted throughout the text. No book on any mathematical discipline can be truly "linear" in its development. Branching in various directions is always possible, frequently desirable and sometimes inevitable. So the pruning shears must be ruthlessly employed! (Indeed, the reader may have preferred a greater degree of ruthlessness!)

Matching theory has not developed in a vacuum. Indeed it has often been in attendance when many of the exciting new concepts in combinatorial optimization have been born. It provided an archetypal minimax theorem which in turn places it close to the birth of duality theory in linear programming. The matching polytope was the first non-trivial polyhedron to be studied by Edmonds as he broke new ground in the areas of facet determination and "good" characterizations. His non-bipartite matching algorithm is a landmark which showed that matching, although certainly not an easy problem, was solvable in polynomial time.

We cannot pursue all these related areas in detail in this book of course. But we have decided to insert boxes of material at various points to provide more background information for the reader. Usually in these boxes we present, in a condensed manner, background material which may be useful to some readers, but which may be well-known to others. For the reader with more background in these topics, we have tried to insert these boxes in such a way that they may be skipped without unduly disrupting the flow of our presentation. At the other extreme, the reader with little — or no — background in these areas will frequently want to read further and we hope that these boxes, together with the references cited, will help in these efforts.

We have also included a brief section on basic terminology, an index of terms and an index of symbols to help the reader translate this book into English! It is likely that no two graph theorists agree on terminology (the two authors certainly do not!), but we hope that you, gentle reader, can learn to live with ours. As for us, we are rather proud to say that our friendship has survived this acid test of terminology selection!

A "pruned" bibliography follows the text. The term "pruned" is used here because in the final analysis we have decided for the sake of space to include only those references on matching which we cite in the text. Many other relevant papers exist, however, and a much more extensive bibliography on matching will be published separately by the authors.

A few remarks are now in order about the format of the book. Most definitions appear in the body of the text and will be set in **bold face** type when they occur for the first time. Sometimes, however, these bold face expressions will appear more than once; for example, when a concept must be recalled from some distant point earlier in the book. All these definitions and others can be found in our index of terms.

Our numbering scheme will be as follows. Theorems, lemmas, corollaries and exercises are all in the same basket as far as numbering is concerned. A string like "$x.y.z.$ Theorem" will refer to theorem z in Section y of Chapter x. Equations and inequalities and a few other displayed strings will be numbered separately, but similarly. Such strings will have three digit number sequences too, but will always be parenthesized. For example, "$(x.y.z)$" will refer to displayed equation (or inequality, etc.) number z in Section y of Chapter x. For further clarity, we endeavor to always say "equation $(x.y.z)$" when referring to this equation in the body of the text and similarly for inequalities, etc.

Finally, figures and tables will be numbered separately from lemmas, theorems, corollaries, exercises, equations, etc. and from each other. The string "Figure $x.y.z$" refers to the zth figure in Section y of Chapter x. Our symbol for the end of a proof (either presented or omitted) of a theorem, lemma or corollary will be the now common mathematical insect called the "black slug" and denoted by "■".

The idea of writing this book probably dates back to some time during 1975-76 when the second author visited Budapest to continue joint research begun with the first author in Nashville in 1972-73. Actually, neither of us quite remembers — or is willing to admit — when such a fatuous idea first occurred! Since that time we have pursued our task — and sometimes each other — in various parts of the world. Hence this peripatetic pair of authors has many people in many places to thank.

This book was done with the TEX editing system on the DEC 1099 computer system at Vanderbilt University. The TEX aspect of this project would never have gotten off the ground without the deep knowledge of "TEXpert" Brendan McKay who gave many hours of his time unselfishly in helping the second author to become acquainted with the system. In addition, McKay designed most of the macros pertinent to the book. Joan McKay typed most of the manuscript into the computer using the TEX system and did so promptly and in a remarkably error-free fashion. We also want to thank Flo Worden and Ruby Moore for some supplemental typing. Maria Perkins and David Palmer of the Vanderbilt Computer Center were most helpful with TEX and other computer related problems. Kathy Goforth has our gratitude for writing the original program to handle the bibliography.

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Basic Terminology

We present here a concise collection of those basic definitions in graph theory we need to get started. Additional terminology will be presented later in the book as needed.

An **undirected graph** (or simply a **graph**) G consists of a finite non-empty set of elements $V(G)$ called **points** and a multi-set of unordered pairs of points $E(G)$ called **lines**. Please note that we are allowing "multiple" or "parallel" lines here, unless otherwise specified. When multiple lines are *not* allowed, we shall call the corresponding graph **simple**. Also, we will not allow **loops** , i.e., lines of the form uu, unless otherwise specified. Unless stated otherwise, p will denote $|V(G)|$ and q will denote $|E(G)|$.

If uv is a line in graph G, line uv is said to **join** points u and v, to be **incident** with points u and v, and points u and v are said to be **adjacent**. Two lines which share a point are also said to be **adjacent**. The set of lines with exactly one endpoint incident with a point in X will be written $\nabla(X)$ and the set of lines with one endpoint in X and the other in Y will be written $\nabla(X,Y)$. The **complement** of a graph G, denoted $\overline{G}$, is that graph having the same point set as G, but in which two such points are adjacent if and only if they are not adjacent in G.

A one-to-one function f mapping $V(G)$ onto $V(H)$ is called an **isomorphism** if f and f^{-1} preserve the number of lines joining each pair of points. If such an isomorphism exists, graphs G and H are said to be **isomorphic**. An isomorphism of graph G onto itself is called an **automorphism** of G. The set of all automorphisms of a graph G under the operation of composition constitute a group called the **automorphism group** of G and denoted by $\mathrm{Aut}(G)$.

The number of lines in graph G incident with a point u is called the **degree** of u (in G) and denoted by $\deg_G(u)$. If graph G is understood, we shall sometimes abbreviate this to $\deg(u)$. A graph in which all degrees are equal to k is said to be **k-regular** and if G is k-regular for some k, we simply say that G is **regular**. A graph which is 3-regular is often called **cubic**.

An alternating sequence of points and lines, beginning and ending with points, is called a **walk**. If all lines in a walk are distinct, the walk is called a **trail**, and if, in addition, the points are also distinct, the trail is a **path**. A family of paths which have no points in common, except possibly their endpoints, will be called **openly disjoint**. The **length of a**

walk is the number of occurrences of lines in it. The **distance** between points u and v, written $d(u, v)$, is the length of any shortest path joining them. If P is a path and u and v are any two points on P, then $P[u, v]$ denotes the subpath of P having endpoints u and v.

A walk or trail in which the first and last points are the same will be said to be **closed**. If a graph contains a closed trail which includes all the lines of G the trail is called an **Euler trail** of G and a graph containing an Eulerian trail is said to be **Eulerian**.

We shall stick by tradition in avoiding the term "closed path", and instead we shall define a **cycle** to be any path of length at least two, together with a line joining the first and last points. The **length** of a cycle will also be the number of lines it contains. A cycle of length n will be called an n**-cycle**. A line joining two points of a cycle, but not itself a line of the cycle, is a **chord** of the cycle. A cycle which includes every point of a graph G is called a **Hamilton** cycle of G. The length of any shortest cycle in a graph G is called the **girth** of G and denoted by $\mathrm{girth}(G)$.

If G is a graph and H is also a graph the points and lines of which are points and lines of G, then H will be called a **subgraph** of G. If H is a subgraph of G and if every line joining two points of H which lies in G also lies in H, we call H an **induced** subgraph of G. If X is a set of points in graph G, then $G[X]$, the subgraph of G **induced by** X, is the induced subgraph of G having point set X. A subgraph H of G is said to be **spanning**, if $V(H) = V(G)$. A spanning subgraph regular of degree n is called an n**-factor**.

A graph in which every pair of points are adjacent is said to be **complete**, and the complete graph on n points is denoted by K_n. A maximal complete subgraph of graph G is called a **clique** of G. A subgraph H is said to be **excluded** with respect to a property $PROP$, if no graph with property $PROP$ has H as a subgraph.

A set of points or lines S in a graph is said to be **minimal** with respect to property $PROP$, if the set has property $PROP$, but no proper subset of S has property $PROP$. Set S is said to be **minimum** with respect to property $PROP$ if, among all subsets of G having property $PROP$, S is one having smallest cardinality The terms **maximal** and **maximum** are defined analogously.

A graph is **connected** if every two points are joined by a path. A maximal connected subgraph of G is called a **component** of G. Components are **even** or **odd** according to whether their point sets have even or odd cardinality.

If the point set of a graph G can be partitioned into two disjoint non-empty sets, $V(G) = A \cup B$, such that all lines of G join a point of A to a point of B, we call G **bipartite** and refer to $A \cup B$ as the **bipartition** of G. In this case we shall also sometimes call the sets A and B the **color classes** of G. A bipartite graph is often also referred to as a **2-colorable** graph or **bigraph.** A special bipartite graph which we shall have occasion to use is $K_{m,n}$, the **complete bipartite graph** having color classes of size m and n and in which every point in each color class is adjacent with every point in the other. In particular, $K_{1,n}$ is called an **n-star** (or sometimes simply, a **star**). A graph containing no cycles is called **acyclic.** An acyclic graph is called a **forest** and if the acyclic graph is also connected, it is called a **tree.** If tree T is a subgraph of graph G and if $V(T) = V(G)$, we call T a **spanning tree** of G.

A set of points S in a connected graph G is a **cutset** if $G - S$ is not connected. A similar definition obtains for a set of lines. If S is a cutset of G consisting of a single point v, the point v is called a **cutpoint** of G, and if S contains a single line e, line e is a **cutline**, or **bridge**, of G. A connected graph containing no cutpoint is called a **non-separable** or **2-connected** graph, or simply a **block.**

If G is not a complete graph, the cardinality of a minimum cutset of points in graph G is called the **(point)-connectivity** of G and is denoted by $\kappa(G)$. If $G = K_n$, κ is defined to be $n-1$. Similarly, the size of a minimum cutset of lines in G is the **line connectivity** of G and is written $\lambda(G)$. A graph G is said to be **k-connected** if $k \leq \kappa(G)$ and to be **k-line-connected** if $k \leq \lambda(G)$. A maximal n-connected subgraph will be called an **n-connected component**, or simply an **n-component.**

A **point coloration** of graph G is an assignment of positive integers to the points of G so that no two points labelled with the same integer are adjacent. G is said to be **n-colorable** if G has a point coloration in n colors. The smallest integer k for which graph G has a coloration of its points in k colors is called the **chromatic number** of G and is denoted by $\chi(G)$. If we assign positive integers to the lines of G so that no two lines with the same integer label are adjacent, we have a **line coloration** of G. The smallest value of k for which G has a line coloration in k colors is called the **chromatic index** of G and is written $\chi_e(G)$.

The **genus** of a graph G, $\gamma(G)$, is the smallest genus of an orientable surface in which G may be embedded so that no two lines meet, except perhaps at their endpoints. Graphs of genus 0 are said to be **planar.**

If the lines of a graph have a direction assigned to them, we have what is known as a "directed graph". More precisely, a **directed graph,** or **digraph**, D consists of a set of **points** $V(D)$ and a set of *ordered* pairs of points $E(D)$ called **lines**. The number of lines having v as their second point is called the **indegree** of v and is denoted by $\deg^{-}(v)$. Similarly, the **outdegree** of point v is the number of lines having v has their first point and is written $\deg^{+}(v)$. The definitions of walk, trail, path and cycle must be modified somewhat in the case of directed graphs. In each of these alternating sequences of points and lines, we shall insist that each (directed) line join the point before it to the point after it *in the sequence*. An **acyclic** digraph is one containing no (directed) cycles. A digraph is **strongly connected** if given every ordered pair of points (u, v), there is a (directed) path from u to v.

A set of lines in a graph G is called **independent** or a **matching** if no two lines have a point in common. The size of any largest matching in G is called the **matching number** of G and is denoted by $\nu(G)$. Now suppose M is a fixed matching in graph G. A point v is said to be **covered**, **matched** or **saturated** by M if some line of M is incident with v. Unmatched points are also called **unsaturated**, **uncovered** or **exposed**. A path (or cycle) P is said to be M**-alternating** if the lines of P are alternately in and not in M. Note that an M-alternating path may begin with a line in M or with a line not in M. If, however, an M-alternating path P begins and ends with lines not in matching M, we call P an M**-augmenting** path. If the matching M is understood, we may simply refer to a path as being **alternating** or **augmenting**. A matching is **perfect** if it covers all of $V(G)$. A graph with a perfect matching is sometimes called a **factorizable** graph.

A **line cover** in a graph G is a set of lines collectively incident with each point of G. The cardinality of any smallest line cover in a graph G is called the **line covering number** of G and is denoted by $\rho(G)$.

A set of points in a graph G is said to be **independent** if no two of them are adjacent. The cardinality of any largest independent set of points in G is known variously as the **(point) independence number** of G, the **stability number** of G and the **vertex packing number** of G, and is written $\alpha(G)$. A set of points S of G is a **point cover** of G if each line of G has at least one endpoint in set S. The cardinality of any smallest point cover is denoted by $\tau(G)$ and is known as the **point covering number** of G.

Given a graph G, the **line graph** of G, $L(G)$, is constructed as follows. The point set of $L(G)$ is $E(G)$ and two points of $L(G)$ are adjacent if and only if they are adjacent as lines in G.

The operation of inserting a new point of degree two on a line of a graph is called **subdividing** the line. If the number of new points of degree two inserted is even, the subdivision is said to be **even**; otherwise it is **odd**.

In this book, the positive integers, the integers, the rationals and the real numbers will be denoted by $\mathcal{Z}_+$, $\mathcal{Z}$, $\mathcal{Q}$ and $\Re$, respectively. The finite field containing two elements will be denoted by $GF(2)$. The greatest integer not greater than real number x will be written as $\lfloor x \rfloor$ and the least integer not less than x as $\lceil x \rceil$. If k is a positive integer, the symbol $k!!$ denotes the product $k(k-2)(k-4)\cdots 4 \cdot 2$ if k is even, and $k(k-2)(k-4)\cdots 3 \cdot 1$ if k is odd.

There are a number of good books on graph theory currently available with terminology mostly consistent with that adopted here. In addition, they offer more extended discussion of these basic concepts as well as a plethora of examples. To mention but two of these books, we direct the reader to the volumes by Bondy and Murty (1976) and Bollobás (1978b).

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The reader should note that, where possible, we have appended a review listing at the end of each reference. If available, we give the *Mathematical Reviews* (MR) code. If that is not available, we provide the code from *Zentralblatt für Mathematik und ihre Grenzgebiete* (Zbl.) and for older entries, we give the code from *Jahrbuch über die Fortschritte der Mathematik* (Jbuch.).

In the case of Russian words, we have adopted the transliteration used by *Mathematical Reviews* when available.

At the end of each entry, the page numbers of the present book upon which the reference is cited are placed in square brackets ([ ]).

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Index of Terms

Those page numbers printed in **bold face** denote the location of the *definition* of the corresponding concept. Note that in some cases, a definition appears in several different places for the convenience of the reader.

Index of Symbols

We list the symbol, the name of the concept symbolized, if any, and then the number of the page upon which the symbol is first defined.

Errata in the book

We are indebted to many colleagues who, over the years, called our attention to errors, inconsistencies, and possible improvements in our book. We are particularly grateful to András Ádam, Chris Godsil, Don Knuth, Charles Little, Gene Lux and Bill Pulleyblank.

Page xiv, line 1: Inequalities should be strict in $0 < p < n$.

Page 2, Exercise 1.0.3(b): Assume that there are no isolated points.

Page 6, line 2: the first "$\Gamma_{G'}(X)$" should be "$\Gamma_G(X)$".

Page 7, Exercise 1.1.8(b): The construction as described does not work. One good definition is

$$\Lambda(X) = X \cup \{a_m + 1\},$$

where $X = \{a_1, \ldots, a_m\}$, $a_0 = 0$, and m is the largest among all indices i, $0 \le i \le k$, for which $2i - a_i$ is maximized. A misprint: $0 \le k < \frac{k}{2}$ should be $0 \le k < \frac{n}{2}$.

Page 12, line -12: "necessarity" should be "necessarily".

Page 14, line -16: "incident" should be "adjacent".

Page 15, line 5: Sentence should be extended to read: "Thus M is a maximum matching and $X \cup Y$ is a minimum cover."

Page 15, lines 13-14: 1.2.1 should be 1.2.2 (twice).

Page 15, line -7: $O(q^2 p)$ should be $O(p^3)$.

Page 20, lines 14-16: "x_0" should be "x".

Page 21, line 9: "bigraph" should be "graph".

Page 21, lines 9-11: Deficiency is supermodular and surplus is submodular (not the other way around).

Page 24, Figure 1.3.1(d): Two diagonal lines from point 3 should be deleted.

Page 31, line -10 (display): G_1 should be $\mathcal{G}_1$.

Page 34, line 7: "of length at least two" should read "having at least two elements".

Page 34, line 13: "lines in" should read "lines of M in".

Page 43, line 14: $v, w \in V(D) - \{s, t\}$ should be $u, v, w \in V(D) - \{s, t\}$.

Page 51, line -14: $j - 1 < l$ should be $j - i < l$.

Page 51, line -2: "e appears in none of $P_{k+1}, \ldots, P_{k+l-1}$" should be "$e$ appears head-to-tail in none of $P_{k+1}, \ldots, P_{k+l-1}$". Also, "$f_{k-1}(e) = f_k(e) = \cdots = f_{k+l-1}(e) \le c(e)$ should be $f_{k-1}(e) \le f_k(e) \le \cdots \le f_{k+l-1}(e) \le c(e)$".

Page 52, line 2: "tail to head" should be "head to tail".

Page 52, Proof of Theorem 2.2.3: Replace the part after "… the bottleneck?" by the following: "Let P_i and P_j $(i < j)$ be two consecutive paths in which x is a bottleneck. If P_i and P_j traverse e in different directions, then by Lemma 2.2.6, $|E(P_j)| \geq |E(P_i)| + 2$. If P_i and P_j traverse e in the same direction, then after augmenting along P_i, e becomes saturated, so to be available for P_j, it must have been traversed in the opposite direction by some path P_k with $i < k < j$, and so by Lemma 2.2.6 again, $|E(P_j)| \geq |E(P_k)| + 2 \geq |E(P_i)| + 4$. Since for each i, $|E(P_i)| \leq p - 1$, it follows that at most $p/2$ of the P_i's can contain x as a bottleneck. There are $q \leq p(p-1)$ different lines in D, and hence there are at most $qp/2 \leq (p^3 - p^2)/2$ paths P_i in our original sequence of augmenting paths $P_1, P_2, \ldots$. This proves Theorem 2.2.3."

Page 56, line -10: The definition of $c_i(u,v)$ should be $c(u,v) - f_i(u,v) + f_i(v,u)$.

Page 63, line 10: "five" should read "two".

Page 63, lines 14-15: Replace "In the first four cases, by submodularity (cf. Section 1.2) or simply by" with "By".

Page 64, Figure 2.3.1: The bottom three of the five figures should be deleted.

Page 65, Figure 2.3.3: The lines joining u and v and x_j and y_j should be deleted.

Page 63, line -10: "pairs" should read "the endpoints of lines".

Page 79, condition (3): l is the constant function 1, not the identity.

Pag 81, line 10: $\delta(f; U_n'') \to 0$ should be $\delta(g; U_n'') \to 0$.

Page 84, line 14: "components of odd" should read "components of $G - S$ of odd".

Page 89, line -9: "Tutte's Theorem" should be replaced by "Berge's formula".

Page 91, line 11: should be $X = X' \cup \{v\}$.

Page 96, line 10: $A(G - u)$ should be $A(G)$.

Page 97, line 18: G_i should be G.

Page 97, line 19: "components of G_i" should read "components G_i".

Page 98, line 2: "=" should be "$\geq$".

Page 99, line 1: "to an arbitrary point of G_w" should be "to at least one arbitrary point of G_w"

Page 99, line 11: M'' should be a *perfect* matching of H.

Page 103, line 20: "points in G" should read "points X in G".

Page 119, line -2: "point-disjoint cycles" should read "point-disjoint odd cycles".

Page 214, line 6: "disjoint" should be "nonadjacent".

Page 267, line 6: "and" should be "the".

Page 287: The conjecture formulated after Exercise 7.4.3 was known to be false, by a counterexample due to Fournier [25].

Page 291, line 4: "has solution" should read "has a solution".

Page 315, line 8: "Conjecture 8.1.8" should be "Conjecture 8.1.9".

Page 324, line -12 (display) and sentence before: This should read "Now the sign of any perfect matching F in the new orientation is

$$\overline{\mathrm{sgn}}(F) = (-1)^{|S \cap F|}\mathrm{sgn}(F) = (-1)^{l(F)+|S \cap F|}."$$

In the next line, $\operatorname{sgn}(F)$ should be $\overline{\operatorname{sgn}}(F)$.

Page 331, Exercise 8.4.2: Delate "If variance … D, then", and replace the displayed formula by

$$\mathsf{E}\big(\det(A_s(\overrightarrow{G}))^2\big) = \sum_{F_1 \neq F_2} 2^{a(F_1,F_2)}.$$

Page 331, Exercise 8.4.3: The quantity to be bounded by a polynomial is $\mathsf{E}\big(\det(A_s(\overrightarrow{G}))^2\big)/\Phi^2(G)$.

Page 348: Exercises 8.6.8 and 8.6.9 should be deleted (they are repetitions of Theorem 5.3.10 and 5.3.13).

Page 350, line -3: "familiy" should be "family".

Page 359, line 16: Modify the definition of F as follows: "Construct a forest F such that every connected component of F contains exactly one point of S, every point of S belongs to exactly one component of F, and for every point v of F which is at an odd distance from a point in S the edge of M incident with v should belong to F.

Page 365, line 8: "vskip 4pt" should be deleted.

Page 386, Figure 10.1.1: The set of eleven points contained in boxes is *not* a Tutte-set as claimed. The reader is invited to find a Tutte-set of nine vertices leaving eleven odd components.

Figure 11.3.1: ν must be 2.

Page 430: The proof of Theorem 11.2.7, Claim 2 contains an error. Here is a sketch of the fix. As in the book, we take a maximum matching M in S and a maximum matching M_i in each S_i so that (11.2.4) is maximum. Then it follows (and not just may be assumed!) that $M_1, M_2 \subseteq \text{Span } M$, just as described.

Consider any element $b \notin \text{Span } M$, and let e.g. $b \in S_1$. Then $M \cup b$ contains a unique circuit C and $M_1 \cup b$ contains a unique circuit C' (since $b \notin \text{Span } M_1$ but $M_1 \cup b$ is not a matching). We claim that $C = C'$. In fact, if $C \neq C'$ then choose $c \in C - C'$ and $c' \in C' - C$. Clearly, $c' \in S_1$; but also $c \in S_1$ by the definition of S_1. Then $M + b - c$ and $M_1 + b - c'$ yield a contradiction.

As in the book, we find a maximum matching N such that $(M_1 \cup M_2) - M \not\subseteq \text{Span } N$. Choose $N \cap M$ maximum subject to this constraint. Clearly $N \not\subseteq \text{Span } M$, and hence there exists a $b \in N - \text{Span } M$. Let, say, $b \in S_1$. Then by the above, b forms the same circuit C with both M and M_1. Now let $b' \in C - N$; then $M' = M + b - b'$ and $M_1' = M_1 + b - b'$ yield a contradiction.

Page 436, Figure 11.3.1: Caption should say $\nu = 2$.

Page 453, line -16: "to τ-critical" should be "to connected τ-critical".

Page 454, Figure 12.1.2: C_5 should be K_5.

Page 478, Figure 12.4.3.: The long heavy horizontal line in the bottom graph should be deleted.

Page 494, line -4: Reference to Zbl. 103, 397 should be added.

Page 543: the symbol $\mathfrak{R}_+$ should be added.

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