DIFFERENTIAL TOPOLOGY

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Preface to the AMS Chelsea Edition

We are deeply grateful to the AMS for reissuing *Differential Topology* as part of its AMS Chelsea Book Series. Our elementary introduction to topology via transversality techniques has managed to stay in print for most of the thirty-six years since its original appearance, and we would like to thank Edward Dunne and his colleagues in Providence for ensuring its continuing availability (knock on wood) for the next thirty-six years. The techniques it highlights have, in some sense, a very 1970's flavor. The quixotic hopes of that decade, that singularity theory and catastrophe theory (of whose catastrophic demise the less said the better) would have a revolutionary impact on physics, chemistry, biology, economics, game theory, and investment strategies in the stock market, have proved largely unfounded. However, we have been pleased to find that our students today are, just as were the students of three decades ago, happy with the visceral, down-to-earth approach to topology espoused by books like ours and Milnor's wonderful *Topology from a Differential Viewpoint*. We hope (again knock on wood) that whatever the fashions in mathematics of the next thirty-six years, this will continue to be the case.

Victor Guillemin
Alan Pollack

Preface

The intent of this book is to provide an elementary and intuitive approach to differential topology. The topics covered are nowadays usually discussed in graduate algebraic topology courses as by-products of the big machinery, the homology and cohomology functors. For example, the Borsuk-Ulam theorem drops out of the multiplicative structure on the cohomology ring of projective space; the Lefschetz theorem comes from Poincaré duality and the Künneth theorem; the Jordan-Brouwer separation theorem follows from Alexander duality; and so on. We have two objections to this big-machinery approach: it obscures the elegant intuitive content of the subject matter, and it gives the student the impression that only big machines can do mathematics. We have attempted to make the results mentioned above and results like them (such as the Gauss-Bonnet theorem, the degree theorem, the Hopf theorem on vector fields) the main topic of our book rather than a mixed bag of interesting examples. In doing so we have abandoned algebraic topology altogether. Our point of view is that these theorems belong in a much more geometric realm of topology, namely intersection theory. Of course, intersection theory, properly done, requires its own apparatus: the transversality theorem. We must confess our sense of vulnerability to the charge that we have replaced one machine with another. Perhaps; *chacun á son goût*. The transversality arguments, it seems to us, can be visualized by the student, something which we feel cannot be honestly said of the singular homology functor.

This book is appropriate for a leisurely first year graduate course. We have also successfully taught a course based on the book to juniors and seniors. For undergraduates we suggest that certain topics be deleted—for example, the discussion of De Rham theory—and that the main emphasis be placed on mod 2 intersection theory, with some of the subjects from Chapter 3 presented mod 2 rather than with orientations. (The reader will notice, incidentally, that the section on De Rham theory is completely independent of the rest of the book. We have avoided using it even in our proof of the degree formula in Chapter 4, Section 8. We prove this formula using Stokes theorem rather than the theory of the top cohomology class, as in the usual treatment.)

The book is divided into four chapters. Chapter 1 contains the elementary theory of manifolds and smooth mappings. We define manifolds as subsets of Euclidean space. This has the advantage that manifolds appear as objects already familiar to the student who has studied calculus in $\mathbf{R}^2$ and $\mathbf{R}^3$; they are simply curves and sufaces generalized to higher dimensions. We also avoid confusing the student at the start with the abstract paraphernalia of charts and atlases. The most serious objection to working in Euclidean space is that it obscures the difference between properties intrinsic to the manifold and properties of its embedding. We have endeavored to make the student aware of this distinction, yet we have not scrupled to use the ambient space to make proofs more comprehensible. (See, for example, our use of the tubular neighborhood theorem in Chapter 2, Section 3). To provide cohesiveness to the elementary material, we have tried to emphasize the "stable" and "generic" quality of our definitions; whether this succeeds in making the basics more palatable, we leave to the reader's judgment.

The last two sections of Chapter 1 deal with Sard's theorem and some applications. Our most important use of Sard is in proving the transversality theorem in Chapter 2, but before doing so we use it to deduce the existence of Morse functions and to establish Whitney's embedding theorem. (Incidentally, it may seem pointless to prove the embedding theorem since our manifolds already sit in Euclidean space. We feel we have just placed the emphasis elsewhere: does there exist a k-dimensional mainfold so pathological that one cannot find a diffeomorphic copy of it inside Euclidean space of specified dimension N? Answer: no, provided $N \geq 2k + 1$.)

Chapter 2 begins by adding boundaries to manifolds. We classify one-manifolds and present Hirsch's proof of the Brouwer fixed-point theorem. Then the transversality theorem is derived, implying that transversal intersections are generic. Transversality permits us to define intersection numbers, and the one-manifold classification shows that they are homotopy invariants. At first we do intersection theorem mod 2, so that the student can become familiar with the topology without worrying about orientations. Moreover, mod 2 theory is the natural setting for the last two theorems of the chapter:

the Jordan-Brouwer separation theorem and the Borsuk-Ulam theorem. In each of the last three chapters we have included a section in which the student himself proves major theorems, with detailed guidance from the text. The Jordan-Brouwer separation theorem is the first of these. We found that our students received this enthusiastically, deriving real satisfaction from applying the techniques they had learned to significantly extend the theory.

In Chapter 3 we reconstruct intersection theory to include orientations. The Euler characteristic is defined as a self-intersection number and shown to vanish in odd dimensions. Next a primitive Lefschetz fixed-point theorem is proved and its use illustrated by an informal derivation of the Euler characteristics of compact surfaces. Translated into the vector field context, Lefschetz implies the Poincaré-Hopf index theorem. In an exercise section using the apparatus of the preceding discussions, the student proves the Hopf degree theorem and derives a converse to the index theorem. Finally, we relate the differential Euler characteristic to the combinational one.

Chapter 4 concerns forms and integration. The central result is Stokes theorem, which we do essentially as M. Spivak does in his *Calculus on Manifolds*. Stokes is used to prove an elementary but, we believe, largely underrated theorem: the degree formula relating integration to mappings. Finally, from this degree formula we derive the Gauss-Bonnet theorem for hypersurfaces in Euclidean space. Chapter 4 also includes an exercise section in which the student can construct De Rham cohomology and prove homotopy invariance. Although other problems relate cohomology to integration and intersection theory, the subject is treated essentially as an interesting aside to our primary discussion. (In particular, cohomology is not referred to anywhere else in the text.)

The original inspiration for this book was J. Milnor's lovely *Topology from the Differential Viewpoint*. Although our book in its present form involves a larger inventory of topics than Milnor's book, our debt to him remains clear.

We are indebted to Dan Quillen and John Mather for the elegant formulation of the transversality theorem given in Chapter 2, Section 3. We are also grateful to Jim King, Isadore Singer, Frank Warner, and Mike Cowan for valuable criticism, to Dennis Sullivan, Shlomo Sternberg, and Jim Munkres for many helpful conversations, and to Rena Themistocles and Phyllis Ruby for converting illegible scribble into typed manuscript. Most particularly, we are indebted to Barret O'Neill for an invaluable, detailed review of our first draft.

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Straight Forward to the Student

This book is written for mathematics students who have had one year of analysis and one semester of linear algebra. Included in the analysis background should be familiarity with basic topological concepts in Euclidean space: openness, connectedness, compactness, etc. We borrow two theorems from analysis which some readers may not have studied: the inverse function theorem, which is used throughout the text; and the change of variable formula for multiple integration, which is needed only for Chapter 4. An excellent reference source for these facts is Spivak [2] (pages 34 and 67 respectively).

The exercises are not only indispensable to understanding the material; they are freely referred to by the text. Those felt to be particularly fundamental—including the ones later referenced—are indicated by asterisks (*). (In particular, asterisks are not signals to warn students of hard problems.)

Here is a miscellany of terms which are used but not defined in the text.

For maps $f: X \longrightarrow Y$ of two sets: *injective* $=$ one-to-one; *surjective* $=$ onto; *bijective* $=$ injective $+$ surjective $=$ one-to-one and onto.

A collection of sets $\{U_\alpha\}$ *covers* a set X if X is contained in the union $\bigcup_\alpha U_\alpha$. An *open cover* (or open covering) of X is a collection of open sets $\{U_\alpha\}$ which covers X. One cover $\{V_\beta\}$ is a *refinement* of another, $\{U_\alpha\}$, if every set V_β is contained in at least one U_α. By the *second axiom of countability*, every open cover $\{U_\alpha\}$ in $\mathbf{R}^n$ has a countable refinement. (*Proof:* Take the collection of all open balls which are contained in some U_α, which have rational radii, and which are centered at points having only rational coordinates.)

If X is a subset contained in $\mathbf{R}^n$, then a subset V of X is (*relatively*) *open* in X if it can be written as the intersection of X with an open subset of $\mathbf{R}^n$: $V = \tilde{V} \cap X$, $\tilde{V}$ open in $\mathbf{R}^n$. If Z is a subset of X, we can also speak of open covers of Z in X, meaning coverings of Z by relatively open subsets of X. Every such cover of Z may be written as the intersection of X with a covering of Z by open subsets of $\mathbf{R}^n$. Since the second axiom of countability is valid for $\mathbf{R}^n$, every open cover of Z relative to X has a countable refinement. (Given $\{U_\alpha\}$, relatively open in X, write $U_\alpha = \tilde{U}_\alpha \cap X$. Then let $\tilde{V}_\beta$ be a countable refinement of $\{\tilde{U}_\alpha\}$ in $\mathbf{R}^n$, and define $V_\beta = \tilde{V}_\beta \cap X$).

Table of Symbols

dim	dimension, p. 4
S^n	n-sphere, p. 4
$\lvert\ \rvert$	absolute value, p. 4
$\times$	product sign, pp. 4, 7
Δ	diagonal, p. 7
graph( )	graph (of a mapping), p. 7
df_x	derivative (of f at x), p. 8
$T_x(X)$	tangent space to X at x, p. 9
$f^{-1}(\ )$	preimage (with respect to f), pp. 21, 27
$O(n)$	orthogonal group, p. 22
$\pitchfork$	"is transversal to," p. 28
$\sim$	"is homotopic to," p. 33
$T(X)$	tangent bundle, p. 50
df	derivative (on the tangent bundle), p. 50
H^k	upper half space (in $\mathbf{R}^k$), p. 57
∂	boundary, p. 57
Int	interior, p. 57
$H_x(X)$	upper half space (in T_x), p. 63
$\vec{n}$	normal vector, pp. 64, 106
Y^ϵ	ϵ neighborhood, p. 69
$N(Y)$	normal bundle (of Y in $\mathbf{R}^n$), p. 71
$N(Z; Y)$	relative normal bundle (of Z in Y), p. 76
$\#$	number of points, p. 77
I_2	mod 2 intersection number, pp. 78, 79
$\deg_2$	mod 2 degree, p. 80
W_2	mod 2 winding number, pp. 86, 88
n_x	normal vector at x, p. 98

$-X$	X with reversed orientation, p. 99
X_0, X_1	boundary components of $I \times X$, p. 99
I	intersection number (oriented) pp. 107, 112, 113
deg	degree (of a mapping), p. 108
χ	Euler characteristic, p. 116
L	global Lefschetz number, p. 119
L_x	local Lefschetz number, p. 121
$\vec{v}$	vector field, p. 132
ind	index (of a vector field), pp. 133, 134
$\phi^*\vec{v}$	pull-back (of a vector field), p. 134
$\overrightarrow{\text{grad}}$	gradient, pp. 140, 177
W	winding number, p. 144
arg	argument, p. 151
V^p	p-fold Cartesian product (of V), p. 153
$\mathfrak{I}^p$	space of all p-tensors, p. 154
$\otimes$	tensor product operation, p. 154
det	determinant, p. 154
ϕ_I	basis vectors for the p-fold tensor product, p. 154
S_p	group of permutations of $\{1, \ldots, p\}$, p. 155
$(-1)^\pi$	sign (of the permutation π), p. 155
T^π	T permuted by π, p. 155
Alt	alternation operation, p. 155
Λ^p	space of alternating p-tensors, p. 156
$\wedge$	wedge product, p. 156
Λ	algebra of alternating tensors, p. 159
A^*	pull-back (of a tensor), p. 159
$f^*\omega$	pull-back (of a form), p. 163
dx_I	basis vectors for p-forms, p. 163
$\int_X$	integral over X, p. 167
$\oint$	line integral, p. 172
$d\omega$	exterior derivative, p. 174
$\sim$	is cohomologous to, p. 179
$H^p(X)$	p-th cohomology group, p. 179
$f^\#$	induced map on cohomology, p. 179
P	P operator, p. 179
$d\arg$	argument form, pp. 188, 192
$d\theta$	angle form, p. 192
$\mathfrak{v}_X$	volume form, p. 194
J_f	Jacobian of f p. 195
$\kappa(x)$	curvature at x, p. 195
$S(a, b)$	rectangular solid, p. 202

Bibliography

As collateral reading, we highly recommend the following two books:

[1] J. MILNOR, *Topology from a Differential Viewpoint.* University of Virginia Press, 1965.

[2] M. SPIVAK, *Calculus on Manifolds.* New York: Benjamin, 1965.

We also recommend that you look at some of the following books, which elaborate on matters we have touched on here. We don't propose that you read these books from cover to cover. The assumption is that you will have a few hours free now and then to find out a little more about ideas that have piqued your interest, not that you will spend a whole semester pursuing any topic. Therefore most of these references are not course books; or, if they are, the relevant sections are explicitly cited. You may find the following rating system helpful.

G = The book is elementary. Prerequisites: only a first course in analysis and linear algebra.
PG = The book is a bit less elementary; a little mathematical sophistication is helpful (e.g., some modern algebra, some algebraic topology).
R = Graduate level mathematics, but a perceptive undergraduate can gain some insights from it.
X = The book is hard going for a graduate student, meant to be read more for inspiration than for comprehension.

Here are some references for the material in Chapter 1.

[3] J. MILNOR, *Morse Theory*. Princeton, N.J.: Princeton University Press, No. 51, 1963.
We showed in Chapter 1, Section 7 that every manifold possesses Morse functions. For the implications of this fact, look at the first part of this book (pp. 1–42). A little algebraic topology is helpful but not indispensable [PG].

[4] M. MORSE, *Pits, Peaks, and Passes*. Produced by the Committee on Educational Media, Mathematical Association of America. Released by Martin Learning Aids, 1966.
In this engaging film, Marston Morse shows how geography tells us about topology. In two dimensions, the critical points of a Morse function correspond to the mountains, valleys, and passes on the graph surface [G].

[5] A. WALLACE, *Differential Topology, First Steps*. New York: Benjamin, 1968.
The author discusses some formidable subjects (Morse theory and surgery) from an elementary point of view and also gives a Morse theoretical classification of two-manifolds [PG].

[6] A. GRAMAIN, *Cours d'initiation à la topologie algébrique*, Orsay, Faculté des Sciences, 1970.
Here is another place where you can find two-manifolds classified using Morse theory. For beginners, this book is a little more pedestrian and easier to read than Wallace. (However, it is written in French.) [G]

[7] L. AHLFORS and L. SARIO, *Riemann Surfaces*. Princeton, N.J.: Princeton University Press, 1960.
Here you can find a classical treatment of the classification of two-manifolds [PG].

[8] R. ABRAHAM, *Transversal Mappings and Flows*. New York: Benjamin, 1967.
The transversality theorem has important applications in the theory of dynamical systems. Before glancing through Abraham's book for this subject, however, you should read the Smale paper cited below [R].

[9] M. GOLUBITSKY and V. GUILLEMIN, *Stable Mappings and Their Singularities*. New York: Springer, 1973. Another interesting application of transversality is presented here: the study of singularities of mappings [R].

Concerning the material in Chapters 2 and 3, we strongly recommend that you read Milnor's book [1]. In particular, Section 7 of Milnor contains an introduction to framed cobordism. If the material interests you, we recommend you pursue the subject further through some of the following sources.

[10] L. PONTRYAGIN, "Smooth Manifolds and Their Applications in Homotopy Theory," *Amer. Math. Society Translations*, Series 2, **II**(1959), 1-114.

This is fairly easy to read; don't be put off by the unbending lemma-theorem-corollary format or by the Slavicisms that have managed to creep into the translation. It is helpful to know a little homotopy theory [R].

[11] P. ALEXANDROFF and H. HOPF, *Topologie*. New York: Chelsea, 1965 (reprint of the original edition published in Berlin in 1935).

One of the classical treatises on topology. It is not easy reading—it is written in German, which will deter some of you—but you will find some delightful applications of the ideas we have discussed [X].

The usual formulation of the Lefschetz Fixed-Point Theorem includes a prescription for computing the global Lefschetz number in terms of homology theory. If you are interested in seeing how this goes, and you've had some algebraic topology, you might look at

[12] M. GREENBERG, *Lectures on Algebraic Topology*. New York: Benjamin, 1967, Section 30 [PG].

If you want to go a little deeper into the topological aspects of flows and vector fields than we did, a good place to start is

[13] W. HUREWICZ, *Lectures on Ordinary Differential Equations*. Cambridge, Mass.: The MIT Press, 1958, Chapter 5, pp. 102-115 [G].

We also recommend a very readable survey article:

[14] S. SMALE, "Differentiable Dynamical Systems," *Bulletin of the A.M.S.*, **73** (1967), 747-817 [PG].

For the material in Chapter 4, our main reference is Spivak [2]. It probably wouldn't do you any harm, however, to go back to a calculus text like Apostol and remind yourself what the usual Green's theorem, Divergence Theorem, and Stokes theorem are.

For the argument principle, look at

[15] AHLFORS, *Complex Analysis*. New York: McGraw-Hill, 1953, p. 123 [PG].

The form of the Gauss-Bonnet theorem we proved in Chapter 4, Section 9 is the so-called "extrinsic" Gauss-Bonnet theorem (for hypersurfaces in $\mathbf{R}^n$). There is a much more subtle "intrinsic" Gauss-Bonnet theorem:

[16] S. CHERN, "A Simple Intrinsic Proof of the Gauss-Bonnet Formula for Closed Riemannian Manifolds," *Annals of Math*, **45** (1944), 747-752 [X].

You'll probably find the Chern paper hard to read unless you have a fairly strong background in differential geometry. A very readable discussion of the two-dimensional intrinsic Gauss-Bonnet theorem can be found, however, in Chapter 7 of

[17] I. M. SINGER and H. A. THORPE, *Lecture Notes on Elementary Geometry and Topology*. Glenview, Ill.: Scott, Foresman, 1967 [PG].

(The reason that the two- dimensional case is simpler than the n-dimensional case is that the group of 2×2 orthogonal matrices is abelian!)

For the relationship between the intrinsic and extrinsic forms of the Gauss-Bonnet theorem, look at Chapter 8 of Singer-Thorpe.

For an introduction to cohomology through forms, as in Section 6, we recommend

[18] M. SPIVAK, *A Comprehensive Introduction to Differential Geometry.* Vol. 1, Boston, Mass.: Publish or Perish, Inc.

Apropos, we highly recommend Spivak as a general textbook. It is the best introduction we know of to graduate level differential geometry. Unfortunately, it is rather hard to obtain. Bookstores don't seem to carry it. You can, however, get it by writing directly to Publish or Perish, Inc., 6 Beacon St., Boston, Mass. 02108.

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