Conformal Invariants

Topics in Geometric Function Theory

Lars V. Ahlfors

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Foreword

Lars Ahlfors often spoke of his excitement as a young student listening to Rolf Nevanlinna's lectures on the new theory of meromorphic functions. It was, as he writes in his collected papers, his "first exposure to live mathematics." In his enormously influential research papers and in his equally influential books, Ahlfors shared with the reader, both professional and student, that excitement.

The present volume derives from lectures given at Harvard over many years, and the topics would now be considered quite classical. At the time the book was published, in 1973, most of the results were already decades old. Nevertheless, the mathematics feels very much alive and still exciting, for one hears clearly the voice of a master speaking with deep understanding of the importance of the ideas that make up the course.

Moreover, several of those ideas originated with or were cultivated by the author. The opening chapter on Schwarz's lemma contains Ahlfors' celebrated discovery, from 1938, of the connection between that very classical result and conformal metrics of negative curvature. The theme of using conformal metrics in connection with conformal mapping is elucidated in the longest chapter of the book, on extremal length. It would be hard to overstate the impact of that method, but until the book's publication there were very few places to find a coherent exposition of the main ideas and applications. Ahlfors credited Arne Beurling as the principal originator, and with the publication of Beurling's collected papers [2] one now has access to some of his own reflections.

Extremal problems are a recurring theme, and this strongly influences the choices Ahlfors makes throughout the book. Capacity is often discussed in relation to small point sets in function theory, with implications for existence theorems, but in that chapter Ahlfors has a different goal, aiming instead for the solution of a geometric extremal problem on closed subsets of the unit circle. The method of harmonic measure appeals to the Euclidean geometry of a domain and parts of its boundary to systematize the use of the maximum principle. Here Ahlfors concentrates on two problems, Milloux's problem, as treated in Beurling's landmark thesis, and a precise version of Hadamard's three circles theorem in a form given by Teichmüller. Nowhere else is there an accessible version of Teichmüller's solution. The chapter on harmonic measure provides only a small sample of a large circle of ideas, developed more systematically in the recent book [7].

Ahlfors devotes four short chapters to discussions of extremal problems for univalent functions, with focus on Loewner's parametric method and Schiffer's variational method. The material on coefficient estimates is now quite dated, following the proof of the Bieberbach conjecture by Louis de Branges [3] and its subsequent adaptation [6] appealing to the classical form of Loewner's differential equation. However, the methods of Loewner and Schiffer have broad applications in geometric function theory and their relevance is undiminished. More detailed treatments have since appeared [8,4], but Ahlfors' overview still brings these ideas to life. In recent years, Loewner's method has stepped into the limelight again with Oded Schramm's discovery of the stochastic Loewner equation and its connections with mathematical physics.

The final two chapters give an introduction to Riemann surfaces, with topological and analytical background supplied to support a proof of the uniformization theorem. In the author's treatment, as in all treatments, the main difficulty is in the parabolic case. Overall, the reader is encouraged to consult other sources for more details, for example [5].

We close with Ahlfors' own words from an address in 1953 at a conference celebrating the centennial of Riemann's dissertation [1]:

> Geometric function theory of one variable is already a highly developed branch of mathematics, and it is not one in which an easily formulated classical problem awaits its solution. On the contrary it is a field in which the formulation of essential problems is almost as important as their solution; it is a subject in which methods and principles are all-important, while an isolated result, however pretty and however difficult to prove, carries little weight.

The reader can learn much of this from the present volume. Furthermore, Ahlfors' remarks came around the time that quasiconformal mappings and, later, Kleinian groups began to flower, fields in which he was the leader. What a second volume those topics would have made!

Peter Duren
F. W. Gehring
Brad Osgood

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Preface

This is a textbook primarily intended for students with approximately a year's background in complex variable theory. The material has been collected from lecture courses given over a long period of years, mostly at Harvard University. The book emphasizes classic and semiclassic results which the author feels every student of complex analysis should know before embarking on independent research. The selection of topics is rather arbitrary, but reflects the author's preference for the geometric approach. There is no attempt to cover recent advances in more specialized directions.

Most conformal invariants can be described in terms of extremal properties. Conformal invariants and extremal problems are therefore intimately linked and form together the central theme of this book. An obvious reason for publishing these lectures is the fact that much of the material has never appeared in textbook form. In particular this is true of the theory of extremal length, instigated by Arne Beurling, which should really be the subject of a monograph of its own, preferably by Beurling himself. Another topic that has received only scant attention in the textbook literature is Schiffer's variational method, which I have tried to cover as carefully and as thoroughly as I know how. I hope very much that this account will prove readable. I have also included a new proof of $|a_4| \leq 4$ which appeared earlier in a *Festschrift* for M. A. Lavrentiev (in Russian).

The last two chapters, on Riemann surfaces, stand somewhat apart from the rest of the book. They are motivated by the need for a quicker approach to the uniformization theorem than can be obtained from Leo Sario's and my book "Riemann Surfaces."

Some early lectures of mine at Oklahoma A. and M. College had been transcribed by R. Osserman and M. Gerstenhaber, as was a lecture at Harvard University on extremal methods by E. Schlesinger. These writeups were of great help in assembling the present version. I also express my gratitude to F. Gehring without whose encouragement I would not have gone ahead with publication.

There is some overlap with Makoto Ohtsuka's book "Dirichlet Problem, Extremal Length and Prime Ends" (Van Nostrand, 1970) which is partly based on my lectures at Harvard University and in Japan.

Lars V. Ahlfors

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ERRATA

Unfortunately, a substantial number of typographical errors escaped detection when the book first went into print. Our thanks to the many colleagues who have made contributions to this list of corrections. We fear, however, that other errors may still have escaped detection. *Caveat lector*!

Page *vi*, lines 16 and 18. Löewner's *should read* Löwner's.

Page 13, Lemma 1-1. $K(\rho) \le 1$ *should read* $K(\rho) \le -1$.

Page 19, equation 1-29. $\dfrac{4 - \log|[f(z)]|}{4 - \log|[f(0)]|}$ *should read* $\dfrac{4 - \log|\zeta(f(z))|}{4 - \log|\zeta(f(0))|}$.

Page 21, line 6. $[58, \dots]$ *should read* $[59, \dots]$.

Page 29, line -4. unit disk *should read* exterior of the unit disk.

Page 29, line -3. comprises *should read* contains.

Page 29, line -2. $1 \ge \frac{1}{4}|b|$ *should read* $1 \ge 1/(4|b|)$.

Page 30, line 4. $\iint_\Omega \left[\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2\right] dxdy$ *should read*

$$\iint_\Omega \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dxdy\,.$$

Page 32, line 1. nonincreasing *should read* nondecreasing.

Page 36, lines 9, 10. [59] and [54] *should read* [61] and [52], respectively.

Page 40, last line. $\{|z| > R\}$ *should read* $\{|z| > R\,, \operatorname{Im} z > 0\}$.

Page 42, line 1. [61] *should read* [6].

Page 43, line 14.

$$\frac{d\log M(r)}{d\log r} \le \frac{4}{\pi\theta(r)} \log M(r) \quad \textit{should read} \quad \frac{d\log M(r)}{d\log r} \ge \frac{4}{\pi\theta(r)} \log M(r)\,.$$

Page 43, line 18. $\log[1 - \omega(r_0)] \le \cdots$ *should read* $1 - \omega(r_0) \le \cdots$.

Page 44, line -3. $\log M(r) \le \dfrac{\log R}{\log r} \log M$ *should read* $\log M(r) \le \dfrac{\log r}{\log R} \log M$.

Page 45, line 8. [60] *should read* [62].

Page 45, Lemma 3-1. its minimum on $|z| = R$ at R, and its maximum on ... *should read* its maximum on $|z| = R$ at R, and its minimum on ... .

Page 46, line 4. $g(\frac{R^m}{M})$ *should read* $g(-\rho, \frac{R^m}{M})$.

Page 46, equation after (3-9).

$$\frac{1}{2\pi}\int_C \cdots - \sum_1^N \omega(a_i) = \quad \textit{should read} \quad \frac{1}{2\pi}\int_C \cdots + \sum_1^N \omega(a_i) = .$$

Page 49, line 1. $|f(0)| \leq \cdots$ *should read* $|f(z_0)| \leq \cdots$.

Page 51, line 13. $L(\Gamma, \rho) = L(\Gamma', \rho')$ *should read* $L(\gamma, \rho) = L(\gamma', \rho')$.

Page 51, Definition 4-1.

$$\frac{\sup_\rho L(\Gamma, \rho)^2}{A(\Omega, \rho)} \qquad \textit{should read} \qquad \sup_\rho \frac{L(\Gamma, \rho)^2}{A(\Omega, \rho)}.$$

Page 67, lines 5,6. $d_\Omega(E_1, E_2) \geq 1/D(u)$ *should read* $d_\Omega(E_1, E_2) \leq 1/D(u)$.

Page 72, line 11. $e^{2\pi M(r)}$ *should read* $e^{2\pi M(R)}$.

Page 74, line -2. p *should read* $\wp$.

Page 77, line 9. E_1' and E_2' ... *should read* $\widehat{E}_1'$ and $\widehat{E}_2'$ ... .

Page 78, Corollary. $\int_a^b dx/\theta(x) \geq \frac{1}{2}$ *should read* $\int_a^b dx/\theta(x) \geq 1$.

Page 79, line 16. $d(C_r, E) \leq \alpha$ *should read* $d(C_r, E) \leq \alpha/2\pi$.

Page 79, line -12. (4-26) *should read* (4-24).

Page 80, line -6. Theorem 4-8 *should read* Theorem 4-9.

Page 80, line -4. $-(1/\pi)\log E$ *should read* $-(1/\pi)\log \operatorname{cap} E$.

Page 81, line -3. Theorems 4-5 and 4-6 *should read* Theorems 4-4 and 4-5.

Page 84, line 1. $g(z) = zh(z^2)^{\frac{1}{2}}$ *should read* $g(z) = zh(z^2)$.

Page 88, inequality (5-11). $|c_1|^3$ *should read* $|c_1|^2$.

Page 88, displayed inequality between (5-12) and (5-13). In the first term on the right-hand side, the denominator $3(1 - |b_1|^2)^2$ *should read* $3(1 - |b_1|^2)$.

Page 90, line 11. Pommerenke [52,53] *should read* Pommerenke [53,54].

Page 101, line 7. $\overline{\Gamma}(z_0, z)\,\frac{\rho^2 e^{i\alpha}}{t - z_0}$ *should read* $\overline{\Gamma}(z_0, z)\,\frac{\rho^2}{t - z_0}$.

Page 103, line -6. $e^{-i\alpha}\left[B(\zeta) + \frac{A(0)}{2}\right]$ *should read* $e^{-i\alpha}\left[B(\zeta) + \frac{\overline{A(0)}}{2}\right]$.

Page 110, line 15. $Q(z) = 0$ *should read* $Q(z) \geq 0$.

Page 146, equation (10-12). $S_r u_\rho - \operatorname{Re}\frac{1}{z}$ *should read* $S_r(u_\rho - \operatorname{Re}\frac{1}{z})$.

Page 146, line -3. (4-13) and (4-14) *should read* (10-13) and (10-14).

Page 153, reference [15]. Minimal" Surfaces *should read* Minimal Surfaces".

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