

ASYMPTOTIC ANALYSIS FOR PERIODIC STRUCTURES

A. BENSOUSSAN
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Preface

In the thirty three years since this book appeared, homogenization, or the theory of partial differential equations with rapidly oscillating coefficients, has flourished. The book has been out of print for many years and many other book-level expositions of various aspects of homogenization have appeared in the meantime. We decided to re-print the book, with minor corrections and bibliographical additions, for two reasons. First, we are very fond of the book since it contains work in our favorite subject, which was done at an early part of our career and has cemented our life-long friendship. Second, we want to pay homage to our senior co-author and mentor Jacques-Louis Lions, who is no longer with us. He introduced us to this field and he was the driving force behind this book, with his own contributions and his enthusiasm for carrying out this endeavor.

We hope that the book will still be useful to those interested in homogenization. We would like to thank all our colleagues with whom we have worked on problems in the area of homogenization in the past. The book was typed in LaTeX by Simon Rubinstein-Salzedo at Stanford. We thank him and appreciate very much his help. We also thank the American Mathematical Society for including the book in their publications.

Alain Bensoussan
George Papanicolaou

June 2011

Bibliography

- [1] Robert A. Adams. *Sobolev spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.
- [2] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. *Comm. Pure Appl. Math.*, 12:623–727, 1959.
- [3] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II. *Comm. Pure Appl. Math.*, 17:35–92, 1964.
- [4] Daljit S. Ahluwalia, Edward L. Reiss, and Stephen E. Stone. A uniform asymptotic analysis of dispersive wave motion. *Arch. Rational Mech. Anal.*, 54:340–355, 1974.
- [5] Michel Artola and Georges Duvaut. Homogénéisation d’une plaque renforcée. *C. R. Acad. Sci. Paris Sér. A-B*, 284(12):A707–A710, 1977.
- [6] Hély Attouch and Yoshio Konishi. Convergence d’opérateurs maximaux monotones et inéquations variationnelles. *C. R. Acad. Sci. Paris Sér. A-B*, 282(9):Ai, A467–A469, 1976.
- [7] R. Azencott and G. Ruget. Mélanges d’équations différentielles et grands écarts à la loi des grands nombres. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 38(1):1–54, 1977.
- [8] I. Babuska. Solution of interface problems by homogenization. *SIAM Journal on Math. Analysis*, 8:923–937, 1977.
- [9] A. Bensoussan and J.-L. Lions. Inéquations quasi variationnelles dépendant d’un paramètre. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 4(2):231–255, 1977.
- [10] A. Bensoussan and J.-L. Lions. *Contrôle impulsif et inéquations quasi variationnelles*, volume 11 of *Méthodes Mathématiques de l’Informatique [Mathematical Methods of Information Science]*. Gauthier-Villars, Paris, 1982.
- [11] A. Bensoussan, J.-L. Lions, and G. Papanicolaou. Perturbations et “augmentation” des conditions initiales. In *Singular perturbations and boundary layer theory (Proc. Conf., École Centrale, Lyon, 1976)*, pages 10–29. Lecture Notes in Math., Vol. 594. Springer, Berlin, 1977.
- [12] A. Bensoussan, J.-L. Lions, and G. Papanicolaou. Homogenization and ergodic theory. In *Probability theory (Papers, VIIIth Semester, Stefan Banach Internat. Math. Center, Warsaw, 1976)*, volume 5 of *Banach Center Publ.*, pages 15–25. PWN, Warsaw, 1979.
- [13] Alain Bensoussan, Jacques-L. Lions, and George C. Papanicolaou. Boundary layers and homogenization of transport processes. *Publ. Res. Inst. Math. Sci.*, 15(1):53–157, 1979.
- [14] Alain Bensoussan, Jacques-Louis Lions, and Georges Papanicolaou. Sur quelques phénomènes asymptotiques d’évolution. *C. R. Acad. Sci. Paris Sér. A-B*, 281(10):Ai, A317–A322, 1975.
- [15] Alain Bensoussan, Jacques-Louis Lions, and Georges Papanicolaou. Homogénéisation, correcteurs et problèmes non-linéaires. *C. R. Acad. Sci. Paris Sér. A-B*, 282(22):Aii, A1277–A1282, 1976.
- [16] Alain Bensoussan, Jacques-Louis Lions, and Georges Papanicolaou. Sur la convergence d’opérateurs différentiels avec potentiel fortement oscillant. *C. R. Acad. Sci. Paris Sér. A-B*, 284(11):A587–A592, 1977.
- [17] Marco Biroli. Sur la G -convergence pour des inéquations quasi-variationnelles. *C. R. Acad. Sci. Paris Sér. A-B*, 284(16):A947–A950, 1977.
- [18] L. Boccardo and I. Capuzzo Dolcetta. G -convergenza e problema di Dirichlet unilaterale. *Boll. Un. Mat. Ital. (4)*, 12(1-2):115–123, 1975.
- [19] Lucio Boccardo and Paolo Marcellini. Sulla convergenza delle soluzioni di disequazioni variazionali. *Ann. Mat. Pura Appl. (4)*, 110:137–159, 1976.

- [20] N. N. Bogoliubov and Y. A. Mitropolsky. *Asymptotic methods in the theory of non-linear oscillations*. Translated from the second revised Russian edition. International Monographs on Advanced Mathematics and Physics. Hindustan Publishing Corp., Delhi, Gordon and Breach Science Publishers, New York, 1961.
- [21] J. F. Bourgat. Numerical experiments of the homogenization method for operators with periodic coefficients. In *Computing methods in applied sciences and engineering (Proc. Third Internat. Sympos., Versailles, 1977)*, I, volume 704 of *Lecture Notes in Math.*, pages 330–356. Springer, Berlin, 1979.
- [22] L. Brillouin. *Wave propagation in periodic structures. Electric filters and crystal lattices*. Dover Publications Inc., New York, N. Y., 1953. 2d ed.
- [23] Luciano Carbone. Γ^- -convergence d'intégrales sur des fonctions avec des contraintes sur le gradient. *Comm. Partial Differential Equations*, 2(6):627–651, 1977.
- [24] Robert Wayne Carroll and Ralph E. Showalter. *Singular and degenerate Cauchy problems*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976. Mathematics in Science and Engineering, Vol. 127.
- [25] Sydney Chapman and T. G. Cowling. *The mathematical theory of nonuniform gases*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, third edition, 1990. An account of the kinetic theory of viscosity, thermal conduction and diffusion in gases, In co-operation with D. Burnett, With a foreword by Carlo Cercignani.
- [26] Stephen Childress. New solutions of the kinematic dynamo problem. *J. Mathematical Phys.*, 11:3063–3076, 1970.
- [27] Ferruccio Colombini. On the regularity of solutions of hyperbolic equations with discontinuous coefficients variable in time. *Comm. Partial Differential Equations*, 2(6):653–677, 1977.
- [28] Ferruccio Colombini and Sergio Spagnolo. Sur la convergence de solutions d'équations paraboliques. *J. Math. Pures Appl. (9)*, 56(3):263–305, 1977.
- [29] R. Courant and D. Hilbert. *Methods of mathematical physics. Vol. II*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1989. Partial differential equations, Reprint of the 1962 original, A Wiley-Interscience Publication.
- [30] E. De Giorgi and S. Spagnolo. Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine. *Boll. Un. Mat. Ital. (4)*, 8:391–411, 1973.
- [31] Luciano de Simon. Sull'equazione delle onde con termine noto periodico. *Rend. Ist. Mat. Univ. Trieste*, 1:150–162, 1969.
- [32] François Delebecque and Jean-Pierre Quadrat. Contribution of stochastic control singular perturbation averaging and team theories to an example of large-scale systems: management of hydropower production. *IEEE Trans. Automatic Control*, AC-23(2):209–222, 1978.
- [33] Jacques Deny and Jacques Louis Lions. Espaces de Beppo Levi et applications. *C. R. Acad. Sci. Paris*, 239:1174–1177, 1954.
- [34] C. Doléans-Dade and P.-A. Meyer. Intégrales stochastiques par rapport aux martingales locales. In *Séminaire de Probabilités, IV (Univ. Strasbourg, 1968/69)*, pages 77–107. Lecture Notes in Mathematics, Vol. 124. Springer, Berlin, 1970.
- [35] M. D. Donsker and S. R. S. Varadhan. Asymptotic evaluation of certain Markov process expectations for large time. I. II. *Comm. Pure Appl. Math.*, 28:1–47; *ibid.* 28 (1975), 279–301, 1975.
- [36] J. L. Doob. *Stochastic processes*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1990. Reprint of the 1953 original, A Wiley-Interscience Publication.
- [37] J. J. Duistermaat. *Fourier integral operators*, volume 130 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1996.
- [38] G. Duvaut and J.-L. Lions. *Les inéquations en mécanique et en physique*. Dunod, Paris, 1972. Travaux et Recherches Mathématiques, No. 21.
- [39] William Feller. *An introduction to probability theory and its applications. Vol. II*. John Wiley & Sons Inc., New York, 1966.
- [40] M. I. Freĭdlin. The Dirichlet problem for an equation with periodic coefficients depending on a small parameter. *Teor. Veroyatnost. i Primenen.*, 9:133–139, 1964.
- [41] M. I. Freĭdlin. Fluctuations in dynamical systems with averaging. *Dokl. Akad. Nauk SSSR*, 226(2):273–276, 1976.
- [42] Avner Friedman. *Partial differential equations of parabolic type*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.

- [43] Avner Friedman. *Stochastic differential equations and applications*. Dover Publications Inc., Mineola, NY, 2006. Two volumes bound as one, Reprint of the 1975 and 1976 original published in two volumes.
- [44] K. O. Friedrichs. Symmetric positive linear differential equations. *Comm. Pure Appl. Math.*, 11:333–418, 1958.
- [45] F. R. Gantmacher. *The theory of matrices. Vols. 1, 2*. Translated by K. A. Hirsch. Chelsea Publishing Co., New York, 1959.
- [46] I. M. Gel'fand. Expansion in characteristic functions of an equation with periodic coefficients. *Doklady Akad. Nauk SSSR (N.S.)*, 73:1117–1120, 1950.
- [47] I. I. Gikhman and A. V. Skorokhod. *Stokhasticheskie differentsialnye uravneniya i ikh prilozheniya*. “Naukova Dumka”, Kiev, 1982.
- [48] I. V. Girsanov. On Ito's stochastic integral equation. *Soviet Math. Dokl.*, 2:506–509, 1961.
- [49] Harold Grad. Singular and nonuniform limits of solutions of the Boltzmann equation. In *Transport Theory (Proc. Sympos. Appl. Math., New York, 1967)*, *SIAM-AMS Proc., Vol. I*, pages 269–308. Amer. Math. Soc., Providence, R.I., 1969.
- [50] Theodore E. Harris. *The theory of branching processes*. Dover Phoenix Editions. Dover Publications Inc., Mineola, NY, 2002. Corrected reprint of the 1963 original [Springer, Berlin; MR0163361 (29 #664)].
- [51] Reuben Hersh. Random evolutions: a survey of results and problems. *Rocky Mountain J. Math.*, 4:443–477, 1974. Based on lectures given by Richard Griego, Reuben Hersh, Tom Kurtz and George Papanicolaou, Papers arising from a Conference on Stochastic Differential Equations (Univ. Alberta, Edmonton, Alta., 1972).
- [52] Kiyosi Itô and Henry P. McKean, Jr. *Diffusion processes and their sample paths*. Springer-Verlag, Berlin, 1974. Second printing, corrected, Die Grundlehren der mathematischen Wissenschaften, Band 125.
- [53] Tosio Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.
- [54] Joseph B. Keller. Corrected Bohr-Sommerfeld quantum conditions for nonseparable systems. *Ann. Physics*, 4:180–188, 1958.
- [55] Joseph B. Keller. Geometrical theory of diffraction. *J. Opt. Soc. Amer.*, 52:116–130, 1962.
- [56] Joseph B. Keller and Robert M. Lewis. Asymptotic methods for partial differential equations: the reduced wave equation and Maxwell's equations. In *Surveys in applied mathematics, Vol. I*, volume 1 of *Surveys Appl. Math.*, pages 1–82. Plenum, New York, 1995.
- [57] Srinivasan Kesavan. Homogénéisation et valeurs propres. *C. R. Acad. Sci. Paris Sér. A-B*, 285(4):A229–A232, 1977.
- [58] W. Kohler and G. C. Papanicolaou. Power statistics for wave propagation in one dimension and comparison with radiative transport theory. *J. Mathematical Phys.*, 14:1733–1745, 1973.
- [59] W. Kohler and G. C. Papanicolaou. Power statistics for wave propagation in one dimension and comparison with radiative transport theory. II. *J. Mathematical Phys.*, 15:2186–2197, 1974.
- [60] S. M. Kozlov. Averaging of differential operators with almost periodic, fast oscillating coefficients. *Dokl. Akad. Nauk SSSR*, 236(5):1068–1071, 1977.
- [61] M. G. Kreĭn and M. A. Rutman. Linear operators leaving invariant a cone in a Banach space. *Amer. Math. Soc. Translation*, 1950(26):128, 1950.
- [62] Heinz-Otto Kreiss. Initial boundary value problems for hyperbolic systems. *Comm. Pure Appl. Math.*, 23:277–298, 1970.
- [63] Thomas G. Kurtz. A limit theorem for perturbed operator semigroups with applications to random evolutions. *J. Functional Analysis*, 12:55–67, 1973.
- [64] L. Landau and E. Lifchitz. *Physique théorique (“Landau-Lifshits”)*. Tome 8. Traduit du Russe. [Translations of Russian Works]. “Mir”, Moscow, 1990. Électrodynamique des milieux continus. [Electrodynamics of continuous media], Second Russian edition revised by Lifchitz [Lifshits] and L. Pitayevski [L. P. Pitaevskii], Translated from the second Russian edition by Anne Sokova.
- [65] Peter D. Lax. Asymptotic solutions of oscillatory initial value problems. *Duke Math. J.*, 24:627–646, 1957.
- [66] Jean Leray. Solutions asymptotiques et groupe symplectique. In *Fourier integral operators and partial differential equations (Colloq. Internat., Univ. Nice, Nice, 1974)*, pages 73–97. Lecture Notes in Math., Vol. 459. Springer, Berlin, 1975.

- [67] Robert M. Lewis. Asymptotic methods for the solution of dispersive hyperbolic equations. In *Asymptotic Solutions of Differential Equations and Their Applications (Proc. Sympos., Math. Res. Center, U.S. Army, Univ. Wisconsin, Madison, Wis., 1964)*, pages 53–107. Wiley, New York, 1964.
- [68] J.-L. Lions. *Équations différentielles opérationnelles et problèmes aux limites*. Die Grundlehren der mathematischen Wissenschaften, Bd. 111. Springer-Verlag, Berlin, 1961.
- [69] J.-L. Lions. Lectures on elliptic partial differential equations. In *Tata Institute of Fundamental Research Lectures on Mathematics, No. 10*, pages iii+130+vi. Tata Institute of Fundamental Research, Bombay, 1967.
- [70] J.-L. Lions and E. Magenes. Problemi ai limiti non omogenei. III. *Ann. Scuola Norm. Sup. Pisa (3)*, 15:41–103, 1961.
- [71] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications. Vol. 1*. Travaux et Recherches Mathématiques, No. 17. Dunod, Paris, 1968.
- [72] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications. Vol. 2*. Travaux et Recherches Mathématiques, No. 18. Dunod, Paris, 1968.
- [73] J.-L. Lions and G. Stampacchia. Variational inequalities. *Comm. Pure Appl. Math.*, 20:493–519, 1967.
- [74] Donald Ludwig. Uniform asymptotic expansions at a caustic. *Comm. Pure Appl. Math.*, 19:215–250, 1966.
- [75] Donald Ludwig and Barry Granoff. Propagation of singularities along characteristics with nonuniform multiplicity. *J. Math. Anal. Appl.*, 21:556–574, 1968.
- [76] Paolo Marcellini and Carlo Sbordone. An approach to the asymptotic behaviour of elliptic-parabolic operators. *J. Math. Pures Appl. (9)*, 56(2):157–182, 1977.
- [77] Paolo Marcellini and Carlo Sbordone. Sur quelques questions de G -convergence et d’homogénéisation non linéaire. *C. R. Acad. Sci. Paris Sér. A-B*, 284(10):A535–A537, 1977.
- [78] Paolo Marcellini and Carlo Sbordone. Homogenization of nonuniformly elliptic operators. *Applicable Anal.*, 8(2):101–113, 1978/79.
- [79] V. G. Markov and O. A. Oleĭnik. On propagation of heat in one-dimensional disperse media. *Prikl. Mat. Meh.*, 39(6):1073–1081, 1975.
- [80] Paul-André Meyer. *Probabilités et potentiel*. Publications de l’Institut de Mathématique de l’Université de Strasbourg, No. XIV. Actualités Scientifiques et Industrielles, No. 1318. Hermann, Paris, 1966.
- [81] Norman G. Meyers. An L^p -estimate for the gradient of solutions of second order elliptic divergence equations. *Ann. Scuola Norm. Sup. Pisa (3)*, 17:189–206, 1963.
- [82] È. Muhamadiev. The invertibility of partial differential operators of elliptic type. *Dokl. Akad. Nauk SSSR*, 205:1292–1295, 1972.
- [83] François Murat. Sur l’homogénéisation d’inéquations elliptiques du 2ème ordre, relatives au convexe $k(\psi_1, \psi_2) = \{v \in H_0^1(\Omega) \mid \psi_1 \leq v \leq \psi_2 \text{ p.p. dans } \Omega\}$, 1976.
- [84] François Murat. Compacité par compensation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 5(3):489–507, 1978.
- [85] Jindřich Nečas. *Les méthodes directes en théorie des équations elliptiques*. Masson et Cie, Éditeurs, Paris, 1967.
- [86] Jacques Neveu. *Bases mathématiques du calcul des probabilités*. Préface de R. Fortet. Deuxième édition, revue et corrigée. Masson et Cie, Éditeurs, Paris, 1970.
- [87] Louis Nirenberg. Remarks on strongly elliptic partial differential equations. *Comm. Pure Appl. Math.*, 8:649–675, 1955.
- [88] Farouk Odeh and Joseph B. Keller. Partial differential equations with periodic coefficients and Bloch waves in crystals. *J. Mathematical Phys.*, 5:1499–1504, 1964.
- [89] R. E. O’Malley, Jr. and C. F. Kung. The singularly perturbed linear state regulator problem. II. *SIAM J. Control*, 13:327–337, 1975.
- [90] G. C. Papanicolaou and R. Burridge. Transport equations for the Stokes parameters from Maxwell’s equations in a random medium. *J. Mathematical Phys.*, 16(10):2074–2085, 1975.
- [91] K. R. Parthasarathy. *Probability measures on metric spaces*. AMS Chelsea Publishing, Providence, RI, 2005. Reprint of the 1967 original.
- [92] Jaak Peetre. Another approach to elliptic boundary problems. *Comm. Pure Appl. Math.*, 14:711–731, 1961.
- [93] Mark A. Pinsky. Multiplicative operator functionals and their asymptotic properties. In *Advances in probability and related topics, Vol. 3*, pages 1–100. Dekker, New York, 1974.

- [94] Yu. V. Prohorov. Convergence of random processes and limit theorems in probability theory. *Teor. Veroyatnost. i Primenen.*, 1:177–238, 1956.
- [95] Giuseppe Pulvirenti. Sulla sommabilità L^p delle derivate prime delle soluzioni deboli del problema di Cauchy-Dirichlet per le equazioni lineari del secondo ordine di tipo parabolico. *Matematiche (Catania)*, 22:250–265, 1967.
- [96] Giuseppe Pulvirenti. Ancora sulla sommabilità L^p delle derivate prime delle soluzioni deboli del problema di Cauchy-Dirichlet. *Matematiche (Catania)*, 23:160–165, 1968.
- [97] Jeffrey B. Rauch and Frank J. Massey, III. Differentiability of solutions to hyperbolic initial-boundary value problems. *Trans. Amer. Math. Soc.*, 189:303–318, 1974.
- [98] Edward L. Reiss. The impact problem for the Klein-Gordon equation. *SIAM J. Appl. Math.*, 17:526–542, 1969.
- [99] G. O. Roberts. Spatially periodic dynamos. *Philos. Trans. Roy. Soc. London Ser. A*, 266:535–558, 1970.
- [100] Enrique Sánchez-Palencia. Comportements local et macroscopique d'un type de milieux physiques hétérogènes. *Internat. J. Engrg. Sci.*, 12:331–351, 1974.
- [101] Carlo Sbordone. Su alcune applicazioni di un tipo di convergenza variazionale. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 2(4):617–638, 1975.
- [102] Carlo Sbordone. Sulla G -convergenza di equazioni ellittiche e paraboliche. *Ricerche Mat.*, 24(1):76–136, 1975.
- [103] L. Schwartz. *Théorie des distributions. Tome I*. Actualités Sci. Ind., no. 1091 = Publ. Inst. Math. Univ. Strasbourg 9. Hermann & Cie., Paris, 1950.
- [104] Laurent Schwartz. *Théorie des distributions. Tome II*. Actualités Sci. Ind., no. 1122 = Publ. Inst. Math. Univ. Strasbourg 10. Hermann & Cie., Paris, 1951.
- [105] M. C. Shen and J. B. Keller. Uniform ray theory of surface, internal and acoustic wave propagation in a rotating ocean or atmosphere. *SIAM J. Appl. Math.*, 28:857–875, 1975.
- [106] R. E. Showalter and T. W. Ting. Asymptotic behavior of solutions of pseudo-parabolic partial differential equations. *Ann. Mat. Pura Appl. (4)*, 90:241–258, 1971.
- [107] I. B. Simonenko. A justification of the method of averaging for abstract parabolic equations. *Dokl. Akad. Nauk SSSR*, 191:33–34, 1970.
- [108] I. B. Simonenko. Justification of the averaging method for the convection problem in a field of rapidly oscillating forces and for other parabolic equations. *Mat. Sb. (N.S.)*, 87(129):236–253, 1972.
- [109] S. L. Sobolev. *Nekotorye primeneniya funktsional'noy analiza v matematicheskoy fizike*. Izdat. Leningrad. Gos. Univ., Leningrad, 1950.
- [110] S. Spagnolo. Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche. *Ann. Scuola Norm. Sup. Pisa (3)* 22 (1968), 571–597; errata, *ibid.* (3), 22:673, 1968.
- [111] Sergio Spagnolo. Sul limite delle soluzioni di problemi di Cauchy relativi all'equazione del calore. *Ann. Scuola Norm. Sup. Pisa (3)*, 21:657–699, 1967.
- [112] Daniel W. Stroock and S. R. S. Varadhan. Diffusion processes with continuous coefficients. I. *Comm. Pure Appl. Math.*, 22:345–400, 1969.
- [113] Daniel W. Stroock and S. R. S. Varadhan. Diffusion processes with continuous coefficients. II. *Comm. Pure Appl. Math.*, 22:479–530, 1969.
- [114] Daniel W. Stroock and S. R. S. Varadhan. Diffusion processes with boundary conditions. *Comm. Pure Appl. Math.*, 24:147–225, 1971.
- [115] Fred D. Tappert. The parabolic approximation method. In *Wave propagation and underwater acoustics (Workshop, Mystic, Conn., 1974)*, pages 224–287. Lecture Notes in Phys., Vol. 70. Springer, Berlin, 1977.
- [116] Luc Tartar. Cours Peccot, 1977.
- [117] Luc Tartar. Compensated compactness and applications to partial differential equations. In *Non Linear Mechanics and Analysis (Heriot-Watt Symposium, Volume IV R. J. Knops Editor)*, pages 136–212. Pitman Research Notes in Mathematics, Vol. 39. Pitman, Boston, 1979.
- [118] S. R. S. Varadhan. Asymptotic probabilities and differential equations. *Comm. Pure Appl. Math.*, 19:261–286, 1966.
- [119] A. D. Ventcel'. Rough limit theorems on large deviations for Markov random processes. I. *Teor. Veroyatnost. i Primenen.*, 21(2):235–252, 1976.
- [120] A. D. Ventcel'. Rough limit theorems on large deviations for Markov random processes. II. *Teor. Veroyatnost. i Primenen.*, 21(3):512–526, 1976.

- [121] A. D. Ventcel' and M. I. Freidlin. Small random perturbations of dynamical systems. *Uspehi Mat. Nauk*, 25(1 (151)):3–55, 1970.
- [122] V. N. Vragov. The mixed problem for a certain class of hyperbolic-parabolic equations. *Dokl. Akad. Nauk SSSR*, 224(2):273–276, 1975.
- [123] Shinzo Watanabe. On stochastic differential equations for multi-dimensional diffusion processes with boundary conditions. *J. Math. Kyoto Univ.*, 11:169–180, 1971.
- [124] Shinzo Watanabe. On stochastic differential equations for multi-dimensional diffusion processes with boundary conditions. II. *J. Math. Kyoto Univ.*, 11:545–551, 1971.
- [125] Shinzo Watanabe and Toshio Yamada. On the uniqueness of solutions of stochastic differential equations. II. *J. Math. Kyoto Univ.*, 11:553–563, 1971.
- [126] Calvin H. Wilcox. Measurable eigenvectors for Hermitian matrix-valued polynomials. *J. Math. Anal. Appl.*, 40:12–19, 1972.
- [127] Calvin H. Wilcox. Theory of Bloch waves. *J. Analyse Math.*, 33:146–167, 1978.
- [128] Toshio Yamada and Shinzo Watanabe. On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ.*, 11:155–167, 1971.
- [129] Erich Zauderer. On a modification of Hadamard's method for obtaining fundamental solutions for hyperbolic and parabolic equations. *J. Inst. Math. Appl.*, 8:8–15, 1971.
- [130] J. M. Ziman. *Principles of the theory of solids*. Cambridge University Press, London, second edition, 1972.

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the 1990s, the number of people who have been employed in the public sector has increased in all countries. The increase has been particularly large in the United States, where the public sector has grown from 10.5% of the total workforce in 1970 to 17.5% in 1995. In the United Kingdom, the public sector has grown from 12.5% of the total workforce in 1970 to 22.5% in 1995.

The increase in the public sector has been driven by a number of factors. One major factor is the growth of the welfare state. In many countries, the welfare state has expanded significantly since the 1970s, leading to a large increase in the number of public employees. Another major factor is the growth of the public sector in the service economy. In many countries, the service economy has grown rapidly since the 1970s, leading to a large increase in the number of public employees.

The increase in the public sector has also been driven by a number of other factors. One major factor is the growth of the public sector in the manufacturing sector. In many countries, the manufacturing sector has grown rapidly since the 1970s, leading to a large increase in the number of public employees. Another major factor is the growth of the public sector in the health care sector. In many countries, the health care sector has grown rapidly since the 1970s, leading to a large increase in the number of public employees.

The increase in the public sector has also been driven by a number of other factors. One major factor is the growth of the public sector in the education sector. In many countries, the education sector has grown rapidly since the 1970s, leading to a large increase in the number of public employees. Another major factor is the growth of the public sector in the social services sector. In many countries, the social services sector has grown rapidly since the 1970s, leading to a large increase in the number of public employees.

The increase in the public sector has also been driven by a number of other factors. One major factor is the growth of the public sector in the transportation sector. In many countries, the transportation sector has grown rapidly since the 1970s, leading to a large increase in the number of public employees. Another major factor is the growth of the public sector in the utilities sector. In many countries, the utilities sector has grown rapidly since the 1970s, leading to a large increase in the number of public employees.

The increase in the public sector has also been driven by a number of other factors. One major factor is the growth of the public sector in the housing sector. In many countries, the housing sector has grown rapidly since the 1970s, leading to a large increase in the number of public employees. Another major factor is the growth of the public sector in the environmental sector. In many countries, the environmental sector has grown rapidly since the 1970s, leading to a large increase in the number of public employees.

The increase in the public sector has also been driven by a number of other factors. One major factor is the growth of the public sector in the cultural sector. In many countries, the cultural sector has grown rapidly since the 1970s, leading to a large increase in the number of public employees. Another major factor is the growth of the public sector in the sports sector. In many countries, the sports sector has grown rapidly since the 1970s, leading to a large increase in the number of public employees.

The increase in the public sector has also been driven by a number of other factors. One major factor is the growth of the public sector in the entertainment sector. In many countries, the entertainment sector has grown rapidly since the 1970s, leading to a large increase in the number of public employees. Another major factor is the growth of the public sector in the media sector. In many countries, the media sector has grown rapidly since the 1970s, leading to a large increase in the number of public employees.

The increase in the public sector has also been driven by a number of other factors. One major factor is the growth of the public sector in the information sector. In many countries, the information sector has grown rapidly since the 1970s, leading to a large increase in the number of public employees. Another major factor is the growth of the public sector in the telecommunications sector. In many countries, the telecommunications sector has grown rapidly since the 1970s, leading to a large increase in the number of public employees.