Asymptotic Analysis for Periodic Structures

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Preface

In the thirty three years since this book appeared, homogenization, or the theory of partial differential equations with rapidly oscillating coefficients, has flourished. The book has been out of print for many years and many other book-level expositions of various aspects of homogenization have appeared in the meantime. We decided to re-print the book, with minor corrections and bibliographical additions, for two reasons. First, we are very fond of the book since it contains work in our favorite subject, which was done at an early part of our career and has cemented our life-long friendship. Second, we want to pay homage to our senior co-author and mentor Jacques-Louis Lions, who is no longer with us. He introduced us to this field and he was the driving force behind this book, with his own contributions and his enthusiasm for carrying out this endeavor.

We hope that the book will still be useful to those interested in homogenization. We would like to thank all our colleagues with whom we have worked on problems in the area of homogenization in the past. The book was typed in LaTeX by Simon Rubinstein-Salzedo at Stanford. We thank him and appreciate very much his help. We also thank the American Mathematical Society for including the book in their publications.

Alain Bensoussan
George Papanicolaou

June 2011

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