# Generalized Functions, Volume 6

## Representation Theory and Automorphic Functions

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# Preface

The classical theory of automorphic functions, created by Klein and Poincaré, was concerned with the study of analytic functions in the unit circle that are invariant under a discrete group of transformations. Since the unit circle can be regarded as a Lobachevskii plane in the Poincaré model, we may say that the classical theory of automorphic functions dealt with the study of functions analytic on the Lobachevskii plane and invariant under a discrete group of motions of the plane.

In the subsequent development of the theory of automorphic functions the papers of Hecke, Siegel, Selberg, and a number of other investigators played an essential part. In particular, papers by Godement, Maass, Roelcke, Peterson, and Langlands cover one or another aspect of the connection between automorphic functions and the theory of groups. Another very interesting direction in the theory of automorphic functions can be found in works of Ahlfors and Bers.

The whole development of the theory of automorphic functions pointed forcefully to the necessity of a group-theoretical approach. Recently many of the ideas of the theory have been linked with arbitrary Lie groups and their discrete subgroups.

The connection between the theory of group representations and the theory of automorphic functions was made particularly precise in the last ten or twenty years, in the context of the development of the theory of infinite-dimensional representations of groups. Although this connection was perceived much earlier (for example, in papers of Klein and Hecke), a true understanding was achieved only after the construction of the theory of infinity-dimensional representations of Lie groups.

One of the first papers to establish this relationship was by Gel'fand and Fomin, in which the concepts of representation theory were linked with the theory of dynamical systems and the theory of automorphic functions. The connection of automorphic functions with dynamical systems already occurs, in essence, in earlier papers of Hopf on dynamical systems.

Apart from the theory of infinite-dimensional representations of Lie groups, which had received a strong impetus in the last twenty years (in papers of Gel'fand and Naimark, Harish-Chandra, Gel'fand and Graev, and others), an important part in the construction of the modern theory of automorphic functions was the creation of the theory of algebraic groups by Chevalley, Borel, Harish-Chandra, Tits, and others.

Perhaps one of the most remarkable ideas that have arisen in recent years is that of the group of adeles. In the process of writing this book the authors have convinced themselves how natural many concepts become when they are applied to the group of adeles and its discrete subgroup of principal adeles.

The book consists of three chapters. In the first chapter we consider problems of representation theory and the theory of automorphic functions connected with a Lie group and a discrete subgroup of it. Although the individual questions of this chapter are of a general character, the main results refer to the group of real matrices of order 2 and its discrete subgroups. In particular, in this chapter we give an account, in the language of representation theory, of the remarkable results of Selberg (Selberg's trace formula).

In the second chapter we construct the theory of representations of the group of matrices of order 2 with elements from an arbitrary locally compact topological field. The well-studied theory of representations of the group of complex matrices and the group of real matrices arises here as a special case. Many facts of representation theory become more conceptual in this general approach. We also mention that the special functions over an arbitrary field, which arise naturally in this theory, are closely related to interesting functions in the theory of numbers (Gauss sums, Kloostermann sums, and others).

The third chapter is devoted to a study of the groups of adeles and the natural homogeneous spaces that arise in connection with these groups. Since it is assumed that the reader is not acquainted with the theory of adeles, the first two sections provide an expository account of the basic ideas of this theory.

With the group of adeles there is connected a remarkable homogeneous space (the space of cosets relative to the subgroup of principal adeles), which has been the main object of study in all papers concerned with adeles. But whereas these papers were devoted to the study of the homogeneous space itself, the computation of its volume (the Tamagawa number), and so forth, we study here the space of functions on this homogeneous space (see **§ 4**, **6**, **7**). From this point of view the fundamental paper of Tate, in which he gives a derivation of the functional equation of the Riemann Zeta-function by means of adeles, can be regarded as an analogue (for the case of matrices of order 1) of the study of representations that we pursue here. Many of our results were also obtained later by other methods by Godement, whose work was very useful in writing **§ 4** of this chapter.

The last three sections are devoted to the beginnings of the general theory for adele groups of an arbitrary algebraic reductive group. A fundamental role in this theory is played by a certain group of automorphisms of the function space that forms a representation of the Weyl group. Symmetry with respect to this group is a veritable key to relations of the type of the functional equation for the Riemann Zeta-function. These automorphisms are closely connected with the so-called horospherical maps. The fact that much of the material in these sections is of quite recent origin inevitably leaves its mark on the character of the exposition itself, which is frequently complicated.

The authors hope, however, that the additional burden the reader assumes in coping with these sections is perhaps compensated by the fact that, if he so wishes, he may participate in the work on these far from completely answered questions.

The book can be read independently of the preceding volumes of the series *Generalized Functions*. However, conceptually it is closely connected with the theory of generalized functions and especially with the contents of volume 5, which deals with analogous problems in other material. It can be regarded as a natural extension of the fifth volume.

The authors are deeply indebted to A. A. Kirillov, who has accepted the arduous task of editing the book and of writing one of the sections (Appendix to Chapter II) in which he expounds his own new results.

Since sending the manuscript to the printers the authors have become acquainted with a preprint of an interesting new paper by Langlands, the material of the Summer School on the Theory of Algebraic Groups, and a paper by Moore. In these papers the reader will find additional information on the material of this book.

I. M. Gel'fand

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# Note

The theory of group representations has given us a new understanding of classical results in the theory of automorphic functions and has made it possible to attack the problems of this theory on a wider scale and obtain a number of new and profound results. The language of the theory of adeles—a recently developed branch of mathematics—plays an important role. The book contains many new ideas and results that have so far been accessible only in mathematical journals. Therefore, the book should appeal to various circles of readers interested in contemporary mathematics. It may be recommended to students in advanced courses, to Ph.D. candidates and to research workers in pure mathematics.

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