Algebras, Lattices, Varieties

Volume I

Ralph N. McKenzie
George F. McNulty
Walter F. Taylor

AMS Chelsea Publishing

American Mathematical Society
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This volume is dedicated to our parents—
Annie Laurie and Milton, Helen and George, Portia and John

Preface

This is the first of four volumes devoted to the general theory of algebras and the closely related subject of lattice theory. This area of mathematics has grown very rapidly in the past twenty years. Not only has the literature expanded rapidly, but also the problems have become more sophisticated and the results deeper. The tendency toward specialization and fragmentation accompanying this growth has been countered by the emergence of new research themes (such as congruence class geometry, Maltsev classification, and congruence classification of varieties) and powerful new theories (such as general commutator theory and tame congruence theory), giving the field a degree of unity it never had before. Young mathematicians entering this field today are indeed fortunate, for there are hard and interesting problems to be attacked and sophisticated tools to be used. Even a casual reader of these volumes should gain an insight into the present-day vigor of general algebra.

We regard an algebra as a nonempty set equipped with a system of finitary operations. This concept is broad enough to embrace many familiar mathematical structures yet retains a concrete character. The general theory of algebras borrows techniques and ideas from lattice theory, logic, and category theory and derives inspiration from older, more specialized branches of algebra such as the theories of groups, rings, and modules. The connections between lattice theory and the general theory of algebras are particularly strong. The most productive avenues to understanding the structure of algebras, in all their diversity, generally involve the study of appropriate lattices. The lattice of congruence relations and the lattice of subalgebras of an individual algebra often contain (in a highly distilled form) much information about the internal structure of the algebra and the essential relations among its elements. In order to compare algebras, it is very useful to group them into varieties, which are classes defined by equations. Varieties can in turn be organized in various ways into lattices (e.g., the lattice of varieties, the lattice of interpretability types). The study of such lattices reveals an extraordinarily rich structure in varieties and helps to organize our knowledge about individual algebras and important families of algebras. Varieties themselves are elementary classes in the sense of logic, which affords an entry to model-theoretic ideas and techniques.

Volume 1 is a leisurely paced introduction to general algebra and lattice theory. Besides the fundamental concepts and elementary results, it contains several harder (but basic) results that will be required in later volumes and a final chapter on the beautiful topic of unique factorization. This volume is essentially self-contained. We sometimes omit proofs, but—except in rare cases—only those we believe the reader can easily supply with the lemmas and other materials that are readily at hand. It is explicitly stated when a proof has been omitted for other reasons, such as being outside the scope of the book. We believe that this volume can be used in several ways as the text for a course. The first three chapters introduce basic concepts, giving numerous examples. They can serve as the text for a one-semester undergraduate course in abstract algebra for honors students. (The instructor will probably wish to supplement the text by supplying more detail on groups and rings than we have done.) A talented graduate student of mathematics with no prior exposure to our subject should find these chapters easy reading. Stiff resistance will be encountered only in §2.4—the proof of the Direct Join Decomposition Theorem for modular lattices of finite height—a tightly reasoned argument occupying several pages.

In Chapter 4, the exposition becomes more abstract and the pace somewhat faster. All the basic results of the general theory of algebras are proved in this chapter. (There is one exception: The Homomorphism Theorem can be found in Chapter 1.) An important nonelementary result, the decomposition of a complemented modular algebraic lattice into a product of projective geometries, is proved in §4.8. Chapter 4 can stand by itself as the basis for a one-semester graduate course. (Nevertheless, we would advise spending several weeks in the earlier chapters at the beginning of the course.) The reader who has mastered Chapters 1–4 can confidently go on to Volume 2 without further preliminaries, since the mastery of Chapter 5 is not a requirement for the later material.

Chapter 5 deals with the possible uniqueness of the factorization of an algebra into a direct product of directly indecomposable algebras. As examples, integers, finite groups, and finite lattices admit a unique factorization. The Jordan normal form of a matrix results from the unique decomposition of the representation module of the matrix. This chapter contains many deep and beautiful results. Our favorite is Bjarni Jónsson's theorem giving the unique factorization of finite algebras having a modular congruence lattice and a one-element subalgebra (Theorem 5.4). Since this chapter is essentially self-contained, relying only on the Direct Join Decomposition Theorem in Chapter 2, a one-semester graduate course could be based upon it. We believe that it would be possible to get through the whole volume in a year's course at the graduate level, although none of us has yet had the opportunity to try this experiment.

Volume 2 contains an introduction to first-order logic and model theory (all that is needed for our subject) and extensive treatments of equational logic, equational theories, and the theory of clones. Also included in Volume 2 are many of the deepest results about finite algebras and a very extensive survey of the results on classifying varieties by their Maltsev properties. Later volumes will deal with such advanced topics as commutator theory for congruence modular varieties, tame congruence theory for locally finite varieties, and the fine structure of lattices of equational theories.

Within each volume, chapters and sections within a chapter are numbered by arabic numerals; thus, §4.6 is the sixth section of Chapter 4 (in Volume 1). Important results, definitions, and exercise sets are numbered in one sequence throughout a chapter; for example, Lemma 4.50, Theorem 4.51, and Definition 4.52 occur consecutively in Chapter 4 (actually in §4.6). A major theorem may have satellite lemmas, corollaries, and examples clustered around it and numbered 1, 2, 3, ... A second sequence of numbers, set in the left-hand margins, is used for a catch-all category of statements, claims, minor definitions, equations, etc (with the counter reset to 1 at the start of each chapter). Exercises that we regard as difficult are marked with an asterisk. (Difficult exercises are sometimes accompanied by copious hints, which may make them much easier.)

The beautiful edifice that we strive to portray in these volumes is the product of many hundreds of workers who, for over fifty years, have been tirelessly striving to uncover and understand the fundamental structures of general algebra. In the course of our writing, we have returned again and again to the literature, especially to the books of Birkhoff [1967], Burris and Sankappanavar [1981], Crawley and Dilworth [1972], Grätzer [1978, 1979], Jónsson [1972], Maltsev [1973], and Pierce [1968].

We wish to thank all of our friends, colleagues, and students who offered support, encouragement, and constructive criticism during the years when this volume was taking shape. It is our pleasure to specifically thank Clifford Bergman, Joel Berman, Stanley Burris, Wanda Van Buskirk, Ralph Freese, Tom Harrison, David Hobby, Bjarni Jónsson, Keith Kearnes, Renato Lewin, Jan Mycielski, Richard Pierce, Ivo Rosenberg, and Constantin Tsinakis. Thanks to Deberah Craig and Burt Rashbaum for their excellent typing. Our editor at Wadsworth & Brooks/Cole, John Kimmel, and Production Editor S. M. Bailey, Designer Victoria Van Deventer, and Art Coordinator Lisa Torri at Brooks/Cole have all taken friendly care of the authors and the manuscript and contributed greatly to the quality of the book. Don Pigozzi's contribution to the many long sessions in which the plan for these volumes was forged is greatly appreciated. We regret that he was not able to join us when it came time to write the first volume; nevertheless, his collaboration in the task of bringing this work to press has been extremely valuable to us.

We gladly acknowledge the support given us during the writing of this volume by the National Science Foundation, the Alexander von Humboldt Foundation, and the Philippine–American Educational Foundation through a Fulbright–Hays grant. Apart from our home institutions, the University of Hawaii, the University of the Philippines, and die Technische Hochschule Darmstadt have each provided facilities and hospitality while this project was underway. Finally, we are deeply grateful for the solid support offered by our wives and children over the past five years.

Ralph N. McKenzie
George F. McNulty
Walter F. Taylor

Contents

Preface to the Chelsea Edition

This is a reprinting of the original 1987 edition of *Algebras, Lattices, Varieties, Volume I*. As we conceived it, this volume together with Volume II, as well as a separate book on the commutator in congruence modular varieties—then soon to appear and now available [Freese and McKenzie 1987]–and one on tame congruence theory [Hobby and McKenzie 1988], would provide a foundation for anyone beginning research in the general theory of algebraic systems. This field and its allied field of lattice theory have experienced even more growth in depth and breadth than we foresaw in 1987. Still, it seems to us that our conception of how to enter this field of mathematics is still sound. So we are pleased that the American Mathematical Society is publishing this Chelsea edition.

You will find here a very short Additional Bibliography. Its first part is devoted to books and reports. Most are directly concerned with topics found in the book in your hands. Two topics we did not cover have found important places in our field: namely the theory of natural dualties–see [Clark and Davey 1998] and [Pitkethly and Davey 2005]—and the theory of quasivarieties, see [Gorbunov 1998]. The report [Shevrin 2017] gives an overview of the activities of just one center in the field, with many pointers to the literature. The second part includes articles that represent high points in the field. These include Ralph McKenzie's solution to Tarski's Finite Basis Problem, J. B. Nation's counterexample to Jónsson's Finite Height Conjecture, and the introduction by Libor Barto and Marcin Kozik of absorbing subuniverses. There are four entries by Keith Kearnes and Ágnes Szendrei (one with Emil Kiss, one with Ross Willard) to represent the work of a leading research team. Also in this part of the Additional Bibliography you will find [Bulatov, Jeavons, and Krokhin 2005], which gives an account of the link between the computational complexity of constraint satisfaction problems and the general theory of algebras. This link was made by Peter Jeavons and his collaborators around 2000. In the ensuing years, this connection to constraint satisfaction problems has driven an explosive growth in research, contributing both on the computational complexity side and on the side of the general theory of algebraic systems.

We have also incuded here a collection of errata.

Ralph N. McKenzie
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ALGEBRAS,
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VOLUME I

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Table of Notation

Set-Theoretical Notation

Our elementary set-theoretical notation is standard. It is described fully in the Preliminaries and only the less commonly encountered features are recalled here.

NOTATION	PAGES	DESCRIPTION
ω	7	the set $\{0,1,2,\ldots\}$ of natural numbers.
$\mathbb{Z}$	7	the set of integers.
$\mathbb{Q}$	7	the set of rational numbers.
$\mathbb{R}$	7	the set of real numbers.
$\mathbb{C}$	7	the set of complex numbers.
$\langle x_0,\cdots,x_{n-1}\rangle$, $\langle f_i\colon i\in I\rangle$	5–6	sequence or function-builder notation
$\bar{x}$	5	the sequence $\langle x_0,x_1,\cdots\rangle$ where the length is specified by the context.
$f\colon A\to B$, $A\xrightarrow{f}B$	6	f is a function from A into B.
$f\colon A\twoheadrightarrow B$	6	f is a function from A onto B.
$f\colon A\rightarrowtail B$	6	f is a one-to-one function from A into B.
$f_a, f(a)$	6	the value of f at a.
$\ker f$	6	the kernel of f.
$\mathrm{a}\mapsto f_a$	6	designates f by assignment—e.g., $k\mapsto k^2$ designates the squaring function.
p_i	22	the i^{th} projection function.
p_i^n	142	the i^{th} n-ary projection function.
$\prod A$, $\prod_I A_i$, $\prod_{i\in I} A_i$	6	the direct product of the system $A=\langle A_i\colon i\in I\rangle$.
B^A	6	the set of all functions from A into B.
$\mathrm{End}\,A$	21	the set of functions from A into A.
$\mathbf{End}\,A$	21	the monoid of functions from A into A.
$\mathrm{Aut}\,A$, $\mathrm{Sym}\,A$	21, 119	the set of permutations of A.
$\mathbf{Aut}\,A$, $\mathbf{Sym}\,A$	21, 119	the group of permutations of A.
$\mathrm{Part}\,A$	112–113	the set of partial functions on A.
$\mathbf{Part}\,A$	112–113	the monoid of partial functions on A.

NOTATION	PAGES	DESCRIPTION
$\operatorname{Bin} A$	112–113	the set of binary relations on A.
$\mathbf{Bin}\, A$	112–113	the monoid of binary relations on A.
$\operatorname{Eqv} A$	8	the set of equivalence relations on A.
$\mathbf{Eqv}\, A$	34	the lattice of equivalence relations on A.
$\alpha \circ \beta$	6	the relational product (composition) of α and β.
$\alpha \circ^n \beta$	196	the n-fold relational product of α with β.
$\alpha^{\cup}$	6	the converse of the relation α.
A/α	8	the factor set of A modulo the equivalence relation α.
a/α	8	the equivalence class of a modulo α.
$a\rho b$, $a \equiv_\rho b$, $a \equiv b \pmod \rho$	8	a and b are related by ρ.
0_A	151	$\{\langle a,a\rangle : a \in A\}$—i.e., the identity relation on A. This is the smallest equivalence relation on A.
1_A	151	$\{\langle a,b\rangle : a,b \in A\}$—i.e., the universal relation on A. This is the largest equivalence relation on A.
Card	315	the class of cardinals.

General Algebraic Notation

Generally, we have used boldface to indicate algebras, or sets equipped with additional structure. For example, $\operatorname{Aut} A$ is the set of all permutations of the set A, $\mathbf{Aut}\, A$ is the group of all permutations of A, and $\operatorname{Aut} \mathbf{A}$ is the set of all automorphisms of the algebra $\mathbf{A}$.

NOTATION	PAGES	DESCRIPTION
$\mathbf{A} = \langle A, F\rangle$ $= \langle A, F_i (i \in I)\rangle$	12 142	the algebra $\mathbf{A}$ with universe A and system $F = \langle F_i : i \in I\rangle$ of basic operations.
$\langle A, \Gamma\rangle$	154	a nonindexed algebra.
$Q^{\mathbf{A}}$	12	the interpretation of the operation symbol Q in the algebra $\mathbf{A}$.
$\mathbf{A} \subseteq \mathbf{B}$	19	$\mathbf{A}$ is a subalgebra of $\mathbf{B}$.
$\operatorname{Sg}^{\mathbf{A}}(X)$	25	the subuniverse of $\mathbf{A}$ generated by X.
$\operatorname{Sub} \mathbf{A}$	19	the set of subuniverses of $\mathbf{A}$.
$\mathbf{Sub}\, \mathbf{A}$	27	the lattice of subuniverses of $\mathbf{A}$.
$S(\mathbf{A})$	252–253	a special subuniverse of $\mathbf{A}^4$.
$C(\mathbf{A})$	296	the center (subuniverse) of $\mathbf{A}$, where $\mathbf{A}$ is an algebra with zero.
$h: \mathbf{A} \to \mathbf{B}$, $\mathbf{A} \xrightarrow{h} \mathbf{B}$	20	h is a homomorphism from $\mathbf{A}$ into $\mathbf{B}$.
$h: \mathbf{A} \rightarrowtail \mathbf{B}$, $\mathbf{A} \overset{h}{\rightarrowtail} \mathbf{B}$	20	h embeds $\mathbf{A}$ into $\mathbf{B}$.
$h: \mathbf{A} \twoheadrightarrow \mathbf{B}$, $\mathbf{A} \overset{h}{\twoheadrightarrow} \mathbf{B}$	20–21	h is a homomorphism from $\mathbf{A}$ onto $\mathbf{B}$.
$h: \mathbf{A} \rightarrowtail\!\!\!\twoheadrightarrow \mathbf{B}$, $h: \mathbf{A} \cong \mathbf{B}$	21	h is an isomorphism from $\mathbf{A}$ onto $\mathbf{B}$.
$\hom(\mathbf{A}, \mathbf{B})$	20	the set of homomorphisms from $\mathbf{A}$ into $\mathbf{B}$.
$\operatorname{End} \mathbf{A}$	21	the set of endomorphisms of $\mathbf{A}$.

NOTATION	PAGES	DESCRIPTION
End A	21	the monoid of endomorphisms of **A**.
Aut **A**	21	the set of automorphisms of **A**.
Aut A	21	the group of automorphisms of **A**.
$h(\mathbf{D}, \mathbf{A})$	319–320	the number of homomorphisms from **D** into **A**.
$m(\mathbf{D}, \mathbf{A})$	319–320	the number of embeddings (= monomorphisms) from **D** into **A**.
$\prod \mathbf{A}$, $\prod_I \mathbf{A}_i$, $\prod_{i \in I} \mathbf{A}_i$	22	the direct product of the system $\mathbf{A} = \langle \mathbf{A}_i : i \in I \rangle$ of algebras.
$\mathbf{A}/\theta$	28	the quotient algebra of **A** modulo the congruence relation θ.
γ/ϕ	150	the quotient of two congruence relations.
B^θ	151	$\{x \in A: x\theta y$ for some $y \in B\}$.
$\theta\|_B$	151	the restriction of θ to B.
Con **A**	28	the set of congruence relations on **A**.
Con A	34	the lattice of congruence relations on **A**.
$\mathrm{Cg}^{\mathbf{A}}(X)$	32	the congruence relation on **A** generated by X.
$\mathrm{Cg}^{\mathbf{A}}(a, b)$	33	the (principal) congruence relation on **A** generated by $\{\langle a, b \rangle\}$.
$C(\alpha, \beta; \eta)$	252	α centralizes β modulo η.
$[\alpha, \beta]$	252	the commutator of the congruences α and β.
$(\alpha : \beta)$	252	the centralizer of β modulo α.
$Z(\mathbf{A})$	250	the center (congruence) of **A**.
$\Delta(\mathbf{A})$	252–253	a special congruence on $\mathbf{A}^2$.
$h = g(f_0, \cdots, f_{k-1})$	142	the composition of a k-ary operation g with k n-ary operations $f_0, \cdots, f_{k-1}$.
Clo **A**	143	the clone of term operations of **A**.
Clo_n **A**	143	the clone of n-ary term operations of **A**.
$\mathbf{Clo}_n$ **A**	148	the algebra of n-ary term operations of **A**.
Pol **A**	144–145	the clone of polynomial operations of **A**.
Pol_n **A**	144–145	the clone of n-ary polynomial operations of **A**.
$(\mathrm{Pol}\,\mathbf{A})\|_X$	156	A set of restrictions to X of polynomial operations of **A**.
$\mathbf{A}\|_X$	156	the algebra induced by **A** on X.
$\sqcap \Gamma$, $\sqcap_{i \in I} \gamma_i$	269 (161)	the product of a system $\Gamma = \langle \gamma_i : i \in I \rangle$ of congruences.
$\alpha \times \beta$	269 (161)	the product of the congruences α and β.
IND(**A**)	279	the family of all independent sets of congruences on **A**.
$\mathbf{A} \sim \mathbf{B}$	270	the algebras **A** and **B** are isotopic.
$\mathbf{A} \sim_{\mathbf{C}} \mathbf{B}$	270	the algebras **A** and **B** are isotopic over the algebra **C**.
$\mathbf{A} \sim_{\mathbf{C}}^{\mathrm{mod}} \mathbf{B}$	273	**A** and **B** are modular-isotopic over **C**.
$\mathbf{A} \sim_{1}^{\mathrm{mod}} \mathbf{B}$	273	**A** and **B** are modular-isotopic in one step.

NOTATION	PAGES	DESCRIPTION
$\mathbf{A} \sim^{\mathrm{mod}} \mathbf{B}$	273	**A** and **B** are modular-isotopic.
$\mathbf{A} \sim^{\mathrm{fin}} \mathbf{B}$	322	**A** and **B** are isotopic over a finite **C**.
$\mathbf{A} \sim^{\mathscr{K}} \mathbf{B}$	322	**A** and **B** are isotopic over a member of $\mathscr{K}$.
$\mathbf{M}_{\mathscr{K}}$	260	the monoid of isomorphism classes of algebras in $\mathscr{K}$ under direct product.
$\mathbf{S}_{\mathscr{K}}$	262	the semiring of isomorphism classes of relational structures in $\mathscr{K}$ under direct product and disjoint union.
f_v, f^u	302	unary operations derived from binary decomposition operations.
$\leq_L$, $\leq_R$, $\leq$, and $\equiv$	305	special relations derived from a binary relational structure.
$\boldsymbol{I}(\mathscr{K})$	219	the class of all isomorphic images of members of $\mathscr{K}$.
$\boldsymbol{H}(\mathscr{K})$	23	the class of all homomorphic images of members of $\mathscr{K}$.
$\boldsymbol{S}(\mathscr{K})$	23	the class of all isomorphic images of subalgebras of members of $\mathscr{K}$.
$\boldsymbol{P}(\mathscr{K})$	23	the class of all isomorphic images of direct products of systems of algebras from $\mathscr{K}$.
$\boldsymbol{P}_{fin}(\mathscr{K})$	222	the class of all isomorphic images of direct products of finite systems of algebras from $\mathscr{K}$.
$\boldsymbol{P}_s(\mathscr{K})$	219	the class of all isomorphic images of subdirect products of systems of algebras from $\mathscr{K}$.
$\boldsymbol{V}(\mathscr{K})$	220	the variety generated by $\mathscr{K}$.
$\mathscr{K}_{\mathrm{si}}$	221	the class of all subdirectly irreducible members of $\mathscr{K}$.
$\mathscr{K}_{\mathrm{fin}}$	222	the class of all finite algebras belonging to $\mathscr{K}$.
Spec $\mathscr{K}$	124	the spectrum of $\mathscr{K}$.
$f_{\mathscr{K}}(n)$	241	the free spectrum function of $\mathscr{K}$.
$\Theta(\mathscr{K})$	228	for a given algebra **A**, the meet of all the kernels of homomorphisms from **A** into algebras belonging to $\mathscr{K}$.
$T_\sigma(X)$	229	the set of terms of type σ over X.
$\mathbf{T}_\sigma(X)$	229	the term algebra of type σ over X.
$\mathbf{F}_{\mathscr{K}}(X)$	230	a free algebra in $\boldsymbol{V}(\mathscr{K})$ with free generating set X.
$\mathbf{F}_{\mathscr{K}}(\kappa)$	230	an algebra $\mathbf{F}_{\mathscr{K}}(\bar{X})$ where $\lvert X \rvert = \kappa$.
$p^{\mathbf{A}}$	232	the interpretation of the term p in the algebra **A**.
$\mathbf{A}, \bar{a} \models p \approx q$	234	the sequence $\bar{a}$ satisfies the equation $p \approx q$ in the algebra **A**.
$\mathbf{A} \models p \approx q$	234	the equation $p \approx q$ is true in **A**.
$\mathscr{K} \models p \approx q$	234	the equation $p \approx q$ is true in every algebra in $\mathscr{K}$.

NOTATION	PAGES	DESCRIPTION
$\mathscr{K} \models \Sigma$	234	every equation in the set Σ is true in every algebra in $\mathscr{K}$.
$\mathbf{\Theta}(\mathscr{K})$	236	the equational theory of $\mathscr{K}$.
$\mathbf{Mod}(\Sigma)$	236	the class of all models of Σ.
$W(X)$	239	the set of all words over X.
$\square$	113	the empty word.
$\mathbf{A}^{\mathrm{D}}$	245	the application of the interpretation operator $^{\mathrm{D}}$ to the algebra $\mathbf{A}$.
$\mathscr{V} \equiv \mathscr{W}$	246	$\mathscr{V}$ and $\mathscr{W}$ are equivalent varieties.
$\mathbf{A} \equiv \mathbf{B}$	246	$\mathbf{A}$ and $\mathbf{B}$ are (term) equivalent algebras.
$\mathbf{A}_\omega$, $\mathbf{A}_{m,k}$, and $\mathbf{A}_{\mathbb{Z}}$	104–106	Certain mono-unary algebras.
SETS	134	The category of sets.
$\hom_{\mathbf{C}}(\mathbf{A}, B)$	133	The collection of morphisms in the category $\mathbf{C}$ from object A to object B.
$\mathbf{Alg}_\rho$	134	The category of all algebras of type ρ.
$\mathbf{CAT}\,\mathscr{V}$	135	The category of all algebras belonging to the variety $\mathscr{V}$.
$\mathbf{Clone}_{\mathbf{C}}\,A$	136	The clone of an object A in a category $\mathbf{C}$.

Notation for Lattices and Ordered Sets

NOTATION	PAGES	DESCRIPTION
$\wedge$, $\vee$	16–17, 36	the lattice operations of meet and join.
$\sup X$, $\bigvee X$, $\bigvee_I x_i$	37, 44	the join (least upper bound) of the set $X = \{x_i\colon i \in I\}$ of elements of an ordered set.
$\inf X$, $\bigwedge X$, $\bigwedge_I x_i$	37, 44	the meet (greater lower bound) of the set $X = \{x_i\colon i \in I\}$ of elements of an ordered set.
$a \prec b$	38	a is covered by b (in an ordered set).
$\mathbf{L}^\partial$	40	the dual of the lattice $\mathbf{L}$.
$I[a,b]$	38	the interval from a to b in an ordered set.
$\mathbf{I}[a,b]$	38	the sublattice with universe $I[a,b]$.
$I(a]$	38	the principal ideal determined by an element a.
$I[a)$	38	the principal filter determined by an element a.
$\mathbf{J(L)}$	85	the ordered set of nonzero join irreducible elements of the lattice $\mathbf{L}$.
$\mathbf{P(L)}$	87	the ordered set of all proper prime ideals of $\mathbf{L}$.
$\mathbf{Iso(L, L')}$	85	the lattice of all isotone maps from the lattice $\mathbf{L}$ into the lattice $\mathbf{L}'$.
$\mathbf{Ord(J)}$	85	the lattice of order ideals of the ordered set $\mathbf{J}$.
Idl L	48	the lattice of ideals of $\mathbf{L}$.
Fil L	48	the lattice of filters of $\mathbf{L}$.
Cvx L	48	the lattice of convex subsets of $\mathbf{L}$.

NOTATION	PAGES	DESCRIPTION
IND(**L**)	66	the family of all directly join independent subsets of **L**.
$a \oplus b$	66	the direct join of a and b.
ϕ_a and ψ_b	56–57	the perspectivity maps.
$I[a,b] \nearrow I[a',b']$	56–57	$I[a,b]$ transposes up to $I[a',b']$.
$I[a,b] \searrow I[a',b']$	56–57	$I[a,b]$ transposes down to $I[a',b']$.
$I[a,b] \nearrow^{w} I[a',b']$	91	$I[a,b]$ transposes weakly up into $I[a',b']$.
$I[a,b] \searrow_{w} I[a',b']$	91	$I[a,b]$ transposes weakly down into $I[a',b']$.
$h(a)$	64	the height of an element a of a lattice.
$d(a)$	65	the dimension of an element a of a lattice with dimension function d.
$\mathbf{M}_3$	39, 79	the five element nondistributive modular lattice.
$\mathbf{N}_5$	39, 55	the five element nonmodular lattice.
$\mathbf{F}_{\mathscr{D}}(3)$	39, 241	the free distributive lattice on three free generators.
$\mathbf{F}_{\mathscr{M}}(3)$	39, 241	the free modular lattice on three free generators.
$\pi = \langle \mathbf{P}, \mathbf{\Lambda} \rangle$, $\mathbf{L}^{\pi}$	207	a projective plane and its associated lattice of subspaces.
$\mathbf{\Gamma} = \langle \mathbf{P}, \mathbf{\Lambda} \rangle$, L^{Γ}	209	a projective geometry and its associated lattice of subspaces.
$\dim \mathbf{\Gamma}$	210	the dimension of a projective geometry $\mathbf{\Gamma}$.
$B^{\mathbf{C}}(X)$	187	the set of cardinalities of finite bases of X with respect to the closure operator **C**.
$B^{\mathbf{A}}(X)$	187	the set of cardinalities of finite bases of X with respect to the closure operator $\mathrm{Sg}^{\mathbf{A}}$, where $X \subseteq A$.

Notation for Groups and Rings

NOTATION	PAGES	DESCRIPTION
N_θ	118	the normal subgroup that is the coset of the unit element of a group modulo the congruence θ on the group.
$[x, y]$	121	the commutator of elements x and y of a group.
$\mathrm{Stab}_K(t)$	132	the stabilizer of t in K.
$\mathbb{Z}_{p^\infty}$	171	the infinite quasi-cyclic group.
$\mathbf{F}_{\mathscr{G}}(X)$, $\mathbf{F}_{\mathscr{G}_r}(X)$	120, 240	the free group generated by the free generating set X.
$\mathscr{N}_n$	122	the variety of n-nilpotent groups.
$\mathscr{B}_n$	122	the (Burnside) variety of groups of exponent n.
ann(**R**), ann(x)	176, 251	the annihilator of the ring **R**; the annihilator of the element x of **R**.

Index of Names

Index of Terms

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List of Errata

Page 15:
The display in the upper half of the page describes the equations that must be added to those for Abelian groups to define the class of (unitary) **R**-modules. The following equations should be added:

$$f_r(0) = 0 \text{ for all } r \in R$$
$$f_0(a) = 0 \text{ for all } a \in M$$

Page 27:
In Exercise 1.13.3 Replace the sentence "Let **A** be an algebra." by "Let **A** be an algebra and let B be a subuniverse of **A**."
Page 35:
In Exercise 1.25.5, change "join" to "union".
Page 46:
On lines 18–19, in Theorem 2.14 add the word "nonempty" so that it reads "... the union of any nonempty collection ...."
Page 53:
The sentence on lines 5–7 needs to be expanded to read "The Dedekind-MacNeille completion of the ordered set of rational numbers is (isomorphic to) the ordered set of real numbers with $+\infty$ and $-\infty$ adjoined."
Exercise 2.23.2 should read as follows:
Prove that the Dedekind-MacNeille completion of the rationals in the open interval $(0,1)$ with their usual order is isomorphic to the closed interval $[0,1]$ of real numbers with its usual order.
Page 60:
At the bottom of the page, replace the long sentence beginning with "**L** is said to be ..." and ending at "is a complemented lattice." by: "A lattice **K** is said to be **complemented** iff **K** is bounded and every element of **K** has a complement; **K** is said to be **relatively complemented** iff every interval $I[a,b]$ in **K**, when construed as a sublattice, is a complemented lattice."
Page 61:
In Exercise 2.35.5, add the word "non-zero" so that it reads "Prove that the non-zero join irreducible elements of a ... "
Page 86:
On the line just below CLAIM 3. Delete "$\rho_a(h)$".
Page 97:
On lines 15–16 add the word "modular" and add two commas, so that it reads " G. Birkhoff [1935b] and K. Menger [1936] had earlier established

that every complemented, finite dimensional, modular lattice is isomorphic to a direct product of simple lattices."

Page 108:

On line 18 change the word "identities" to "equations" so that it reads "Although we asserted above that the unary equations ... "

Page 111:

On line 10 add the words "finite, nontrivial" so that it reads, "Prove that for every finite, nontrivial, unary algebra **A** there exists ..."

Page 121:

On line 4 change "G" to "$\mathbf{F}_{\mathcal{G}}(X)$".

Page 132:

Near the end of Exercise 3.15.5, replace "$\psi(K)$" by $\psi(G)$".

Page 137:

On line 27 change period to question mark: "... for some **A** in a given class $\mathcal{K}$ of algebras? (For instance, ... "

Also, on the last line of page 137, replace "$e \cdot x$" and "$f \cdot x$" by "$x \cdot e$" and "$x \cdot f$", respectively.

Page 143:

On line 18 remove one of the two dots at the end of this sentence.

Page 149:

Near end of Exercise 4.9.13, the sentence should read "... then the set of maps of **Clone A**, whose co-domain is **A**, is identical with the centralizer of Clo **A** in Clo A."

Page 155:

In **THE CONGRUENCE GENERATION THEOREM,** the "$Pol_1\mathbf{A}$" should be "$\mathrm{Pol}_1\,\mathbf{A}$".

Page 155:

On lines -12, -13 change "An equivalent condition is that" to "If **A** is finite, an equivalent condition is that"

Page 181:

In the displayed formula on line 8, replace "$\langle b, c\rangle \in H$" by "$\langle b, c\rangle \in H^2$".

Page 195:

In Exercise 4.64.9, replace "for $x > 0$" by "for all x".

Page 202:

On the last line of the page replace "1-element" by "one-element" to achieve consistency within the sentence.

Page 206:

In Exercise 4.77.12, delete "(Theorem 2.68 may be useful.)"

Page 230:

In part **(iii)** of Theorem 4.117, replace the entire statement by "*If* $X \cap I = \emptyset$*, then if* $\mathcal{V}$ *has a nontrivial member there exists an algebra free in* $\mathcal{V}$ *over the set* X."

Page 238:

Definition 4.133 (lines 18–19) should be restated as follows:

DEFINITION 4.133. We define $q^{\mathbf{A}}$, where $q \in \mathbf{F}_{\mathcal{V}}(n)$, $\mathbf{A} \in \mathcal{V}$ and $\mathcal{V}$ is a variety, as follows. Let $x_0, \ldots, x_{n-1}$ be the free generators of $\mathbf{F}_{\mathcal{V}}(n)$ and let $q = p(x_0, \ldots, x_{n-1})$ where $p \in \mathbf{T}_\sigma(n)$. Then we put $q^{\mathbf{A}} = p^{\mathbf{A}}$.

Page 239:

In the last sentence before the line of bold type, at the middle of the page, replace “three” by “two”.

Page 248:

The equations listed in Theorem 4.144 that provide a Maltsev characterization of varieties that are congruence distributive are not actually those framed by Bjarni Jónsson in the paper cited. The difference is that the two cases when i is even and when it is odd have been reversed. Theorem 4.144, in effect, provides a second Maltsev characterization of congruence distributive varieties.

Page 250:

On lines -8, -9 change “R. Freese and R. McKenzie [forthcoming]” to “R. Freese and R. McKenzie [1987].”

Page 333:

The two words in the roman typeface, “countable and” in **COROLLARY 2** should be italicized like the rest of the statement.

Page 343:

On line 14 we are defining 0_A which first occurs on page 151. Replace “$\{\langle a, a\}$:” by “$\{\langle a, a\rangle$:”.

Page 357:

On line 14 in the left-hand column, remove “defined, 16”.

This book presents the foundations of a general theory of algebras. Often called "universal algebra," this theory provides a common framework for all algebraic systems, including groups, rings, modules, fields, and lattices. Each chapter is replete with useful illustrations and exercises that solidify the reader's understanding.

The book begins by developing the main concepts and working tools of algebras and lattices, and continues with examples of classical algebraic systems like groups, semigroups, monoids, and categories. The essence of the book lies in Chapter 4, which provides not only basic concepts and results of general algebra, but also the perspectives and intuitions shared by practitioners of the field. The book finishes with a study of possible uniqueness of factorizations of an algebra into a direct product of directly indecomposable algebras.

There is enough material in this text for a two semester course sequence, but a one semester course could also focus primarily on Chapter 4, with additional topics selected from throughout the text.

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