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Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach

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http://dx.doi.org/10.1090/cln/003

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3 Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach

Courant Institute of Mathematical Sciences
New York University
New York, New York

American Mathematical Society
Providence, Rhode Island

2000 *Mathematics Subject Classification.* Primary 30–XX, 33–XX, 60–XX, 15A90, 26Cxx.

Library of Congress Cataloging-in-Publication Data

Deift, Percy, 1945–
Orthogonal polynomials and random matrices : a Riemann-Hilbert approach / Percy Deift.
p. cm. — (Courant lecture notes ; 3)
Originally published: New York : Courant Institute of Mathematical Sciences, New York University, c1999.
Includes bibliographical references.
ISBN 0-8218-2695-6
1. Orthogonal polynomials. 2. Random matrices. I. Title. II. Series.
QA404.5 .D37 2000
515′.55—dc21 00-061834

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10 9 8 7 6 5 4 3 2 05 04 03 02

To Rebecca and Abby
for your patience and support

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Preface

In the academic year 1996–1997, I gave a course at the Courant Institute on Riemann-Hilbert problems, orthogonal polynomials, and random matrix theory. The lectures for the course were taken down and organized into note form by Randall Pyke, John Podesta, José Ramírez, and Wen-qing Xu. Over the last year, Jinho Baik, Thomas Kriecherbauer, and Ken McLaughlin have helped me further to bring these notes into their present form. Without their help, these notes would never have been published, and I am truly thankful to all these people for their efforts.

I gave the course in 1996–1997 in an attempt to understand from a more rigorous mathematical point of view various results and formulae in Mehta's wonderful book *Random Matrices* [**43**]. At the same time, I was stimulated and challenged by a set of questions from Peter Sarnak, who himself was trying to understand [**43**]. These notes are in many ways a response to his questions, and I deeply appreciate his clear insights and ready help.

The central question is the following: Why do very general ensembles of random $n \times n$ matrices exhibit universal behavior as $n \to \infty$?

My work and that of my collaborators Thomas Kriecherbauer, Ken McLaughlin, Stephanos Venakides, and Xin Zhou on this question is reported in [**15, 16, 17**]. Apart from certain additional preparatory material, these notes are a pedagogic illustration of the general methods and results in [**15, 16, 17**], in a special case (see Sections 7 and 8) in which the technical difficulties are at a minimum. I thank my colleagues for allowing me to reproduce these results here. Pioneering mathematical work on universality for random matrix ensembles was done by Pastur and Scherbina in [**51**], and Its and Bleher in [**5**].

In addition to the students and colleagues mentioned above, I would like to thank Daisy Calderon for her skill and patience in typing the final manuscript. Special thanks are also due to Melissa Macasieb for her expert copy-editing of the text, and to Melissa and Reeva Goldsmith for their care in correcting the TeX file. The final figures were drawn by Daisy Calderon. The entire project of preparing the manuscript for publication was overseen by Paul Monsour, and many, many thanks are due to him for his great expertise and all his help.

This work was supported in part by NSF Grant DMS–9500867.

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Bibliography

[1] Abramowitz, M., and Stegun, I. A., eds. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. Dover, New York, 1972.

[2] Akhiezer, N. I. *The classical moment problem and some related questions in analysis*. Translated by N. Kemmer. Hafner, New York, 1965.

[3] Beals, R., and Coifman, R. Scattering and inverse scattering for first order operators. *Comm. Pure Appl. Math.* 37: 39–90, 1984.

[4] Beals, R., Deift, P., and Tomei, C. *Direct and inverse scattering on the line*. Mathematical Surveys and Monographs, 28. American Mathematical Society, Providence, R.I., 1988.

[5] Bleher, P., and Its, A. R. Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problems, and universality in the matrix model. *Ann. of Math. (2)* 150: 185–266, 1999.

[6] Clancey, K., and Gohberg, I. *Factorization of matrix functions and singular integral operators*. Operator Theory: Advances and Applications, 3, Birkhäuser, Basel–Boston, 1981.

[7] Coddington, E. A., and Levinson, N. *Theory of ordinary differential equations*. McGraw-Hill, New York–Toronto–London, 1955.

[8] Cohen, J., Kesten, H., and Newman, C., eds. *Random matrices and their applications*. Contemporary Mathematics, 50. American Mathematical Society, Providence, R.I., 1984.

[9] Deift, P., Integrable Hamiltonian systems. *Dynamical systems and probabilistic methods in partial differential equations (Berkeley, CA, 1994)*, 103–138. Lectures in Applied Mathematics, 31. American Mathematical Society, Providence, R.I., 1996.

[10] Deift, P. A., Its, A. R., and Zhou, X. A Riemann-Hilbert approach to asymptotic problems arising in the theory of random matrix models and also in the theory of integrable statistical mechanics. *Ann. of Math. (2)* 146: 149–235, 1997.

[11] Deift, P., Kamvissis, S., Kriecherbauer, T., and Zhou, X. The Toda rarefaction problem. *Comm. Pure Appl. Math.* 49: 35–83, 1996.

[12] Deift, P., Li, L. C., and Tomei, C. Toda flows with infinitely many variables. *J. Funct. Anal.* 64: 358–402, 1985.

[13] Deift, P., and McLaughlin, K. T-R. A continuum limit of the Toda lattice. *Mem. Amer. Math. Soc.* 131(624), 1998.

[14] Deift, P., McLaughlin, K. T-R., and Kriecherbauer, T. New results on the equilibrium measure for logarithmic potentials in the presence of an external field. *J. Approx. Theory* 95: 388–475, 1998.

[15] Deift, P., McLaughlin, K. T-R., Kriecherbauer, T., Venakides, S., and Zhou, X. Asymptotics for polynomials orthogonal with respect to varying exponential weights. *Internat. Math. Res. Notices* 16: 759–782, 1997.

[16] ———. Strong asymptotics of orthogonal polynomials with respect to exponential weights. *Comm. Pure Appl. Math.* 52: 1491–1552, 1999.

[17] ———. Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. *Comm. Pure Appl. Math.* 52: 1335–1425, 1999.

[18] Deift, P. A., Nanda, T., and Tomei, C. Ordinary differential equations and the symmetric eigenvalue problem. *SIAM J. Num. Anal.* 20(1): 1–22, 1983.

[19] Deift, P., Venakides, S., and Zhou, X. The collisionless shock region for the long-time behavior of solutions of the KdV equation. *Comm. Pure Appl. Math.* 47(2): 199–206, 1994.

[20] ______. New results in small dispersion KdV by an extension of the steepest descent method for Riemann-Hilbert problems. *Internat. Math. Res. Notices* 6: 286–299, 1997.

[21] Deift, P., and Zhou, X. A steepest descent method for oscillatory Riemann-Hilbert problem, Asymptotics for the MKdV equation. *Ann. of Math.(2)* 137: 295–368, 1993.

[22] ______. Asymptotics for the Painlevé II equation. *Comm. Pure Appl. Math.* 48: 277–337, 1995.

[23] des Cloizeaux, J., and Mehta, M. L. Asymptotic behavior of spacing distributions for the eigenvalues of random matrices. *J. Math. Phys.* 14: 1648–1650, 1973.

[24] Durrett, R. *Probability. Theory and examples.* The Wadsworth & Brooks/Cole Statistics/Probability Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, Calif., 1991.

[25] Dym, H., and McKean, H. P. *Gaussian processes, function theory, and the inverse spectral problem.* Probability and Mathematical Statistics, 31. Academic, New York–London, 1976.

[26] Dyson, F. Fredholm determinants and inverse scattering problems. *Comm. Math. Phys.* 47: 171–183, 1976.

[27] Flaschka, M. The Toda lattice. I. Existence of integrals. *Phys. Rev. B (3)* 9: 1924–1925, 1974.

[28] Fokas, A. S. A unified transform method for solving linear and certain nonlinear PDEs. *Proc. Roy. Soc. London Ser. A* 453: 1411–1443, 1997.

[29] Fokas, A. S., Its, A. R., and Kitaev, A. V. An isomonodromy approach to the theory of two-dimensional quantum gravity. (Russian) *Uspekhi Mat. Nauk* 45(6(276)): 135–136, 1990; translation in *Russian Math. Surveys* 45(6): 155–157, 1990.

[30] ______. Discrete Painlevé equations and their appearance in quantum gravity. *Comm. Math. Phys.* 142(2): 313–344, 1991.

[31] Gohberg, I., and Kreĭn, M. Systems of integral equations on a half-line with kernels depending on the difference of arguments. *Amer. Math. Soc. Transl. (2)* 14: 217–287, 1960.

[32] Hellinger, E. Zur Stichtjesschen Kettenbruchtheorie. *Ann. of Math.* 86: 18–29, 1992.

[33] Johansson, K. On fluctuations of eigenvalues of random Hermitian matrices. *Duke Math. J.* 91(1): 151–204, 1998.

[34] Kamvissis, S. On the long-time behavior of the double infinite Toda lattice under initial data decaying at infinity. *Comm. Math. Phys.* 153(3): 479–519, 1993.

[35] Katznelson, Y. *An introduction to harmonic analysis.* Second corrected edition. Dover, New York, 1976.

[36] Katz, N., and Sarnak, P. Katz, N. M.; Sarnak, P. Zeroes of zeta functions and symmetry. *Bull. Amer. Math. Soc. (N.S.)* 36(1): 1–26, 1999.

[37] Khinchin, A. Ya. *Continued fractions.* University of Chicago Press, Chicago, 1964.

[38] Khovanskii, A. N. *The application of continued fractions and their generalizations to problems in approximation theory.* Translated by Peter Wynn. Noordhoff, Groningen, 1963.

[39] Kline, M. *Mathematical thought from ancient to modern times.* Oxford University Press, New York, 1972.

[40] Landkof, N. S. *Foundations of modern potential theory.* Translated from the Russian by A. P. Doohovskoy. Die Grundlehren der mathematischen Wissenschaften, Band 180. Springer, New York–Heidelberg, 1972.

[41] Lax, P. D., and Levermore, C. D. *The small dispersion limit of the Korteweg-de Vries equation.* I. II. III. *Comm. Pure Appl. Math.* 36: 253–290, 1983; 571–593; 809–829.

[42] Manakov, S. V. Complete integrability and stochastization of discrete dynamical systems. *Soviet Phys. JETP* 40(2): 269–274, 1974.

[43] Mehta, M. L. *Random matrices.* Second edition. Academic, Boston, 1991.

[44] Moser, J. Finitely many mass points on the line under the influence of an exponential potential—an integrable system. *Dynamical systems, theory and applications (Rencontres, BattelleRes. Inst., Seattle, Wash., 1974)*, 467–497. Lecture Notes in Physics, 38. Springer, Berlin, 1975.

[45] Muskhelishvili, N. I. *Singular integral equations. Boundary problems of function theory and their application to mathematical physics.* Translation by J. R. M. Radok. Noordhoff, Groningen, 1953.

[46] Noble, B. *Methods based on the Wiener-Hopf technique for the solution of partial differential equations*. International Series of Monographs on Pure and Applied Mathematics, 7. Pergammon Press, New York–London–Paris–Los Angeles, 1958.

[47] Odlyzko, A. On the distribution of spacings between zeros of the zeta function. *Math. Comp.* 48(177): 273–308, 1987.

[48] Parlett, B. *The Symmetric Eigenvalue Problem*. Prentice-Hall, Englewood Cliffs, N.J., 1980.

[49] Porter, C. E., ed. *Statistical theories of spectra: Fluctuations, A Collection of Reprints, Original Papers, with an Introductory Review*. Academic Press, New York, 1965.

[50] Pastur, L., and Figotin, A. *Spectra of random and almost periodic operators*. Springer-Verlag, Berlin, 1992.

[51] Pastur, L., and Scherbina, L. *Universality of the local eigenvalue statistics for a class of unitary invariant random matrix ensembles*. Preprint, 1996.

[52] Reed, M., and Simon, B. *Methods of modern mathematical physics*, I–IV. Academic Press, New York, 1975–1980.

[53] Royden, H. *Real analysis*. Third edition. Macmillan, New York, 1988.

[54] Rudnick, Z., and Sarnak, P. *Zeros of principal L-functions and random matrix theory*. Princeton University, preprint, 1995.

[55] Ryll-Nardzewski, C., *On the ergodic theorems II: Ergodic theory of continued fractions*. *Studia Math.* 12: 74–79, 1951.

[56] Šabat, A. B. *One-dimensional perturbations of a differential operator, and the inverse scattering problem*, 279–296, 298. *Problems in mechanics and mathematical physics*. Izdat. "Nauka", Moscow, 1976.

[57] Saff, E. B., and Totik, V. *Logarithmic potentials with external fields*. Springer, New York–Berlin, 1997.

[58] Simon, B. *Trace ideals and their applications*. Cambridge University Press, Cambridge–New York, 1974.

[59] ______. The classical moment problem as a self-adjoint finite difference operator. *Adv. in Math.*, in press.

[60] Stein, E. M. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, 30. Princeton University Press, Princeton, N.J., 1970.

[61] Szegö, G. *Orthogonal polynomials*. Fourth edition. American Mathematical Society, Colloquium Publications, 23. American Mathematical Society, Providence, 1975.

[62] Torchinsky, A. *Real variables*. Addison-Wesley, Redwood City, Calif., 1988.

[63] Venakides, S. The zero dispersion limit of the Korteweg-de Vries equation for initial potentials with nontrivial reflection coefficient. *Comm. Pure Appl. Math.* 38: 125–155, 1985.

[64] Venakides, S., Deift, P., and Oba, R. The Toda shock problem. *Comm. Pure Appl. Math.* 44: 1171–1242, 1991.

[65] Wall, H. S. *Analytic theory of continued fractions*. Van Nostrand, New York, 1948.

[66] Warner, F. W. *Foundations of differentiable manifolds and Lie groups*. Scott, Foresman and Company, Glenview, 1971.

[67] Weyl, H. Über gewöhnlike Differential gleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen. *Ann. of Math.* 68: 220–269, 1910.

[68] Widom, H. The asymptotics of a continuous analogue of orthogonal polynomials. *J. Approx. Theory* 77(1): 51–64, 1994.

[69] ______. Asymptotics for the Fredholm determinant of the sine kernel on a union of intervals. *Comm. Math. Phys.* 171(1): 159–180, 1995.

[70] Zakharov, V. E., and Shabat, A. B. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Soviet Phys. JETP* 34(1): 62–69, 1972.

Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach

PERCY DEIFT

This volume expands on a set of lectures held at the Courant Institute on Riemann-Hilbert problems, orthogonal polynomials, and random matrix theory. The goal of the course was to prove universality for a variety of statistical quantities arising in the theory of random matrix models. The central question was the following: Why do very general ensembles of random $n \times n$ matrices exhibit universal behavior as $n \to \infty$? The main ingredient in the proof is the steepest descent method for oscillatory Riemann-Hilbert problems.

ISBN 0-8218-2695-6

9 780821 826959

CLN/3

NEW YORK UNIVERSITY