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Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities

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To Isabelle

Contents

Preface

These notes deal with the theory of Sobolev spaces on Riemannian manifolds. Though Riemannian manifolds are natural extensions of Euclidean space, the naive idea that what is valid for Euclidean space must be valid for manifolds is false. Several surprising phenomena appear when studying Sobolev spaces on manifolds. Questions that are elementary for Euclidean space become challenging and give rise to sophisticated mathematics, where the geometry of the manifold plays a central role. The reader will find many examples of this in the text.

These notes have their origin in a series of lectures given at the Courant Institute of Mathematical Sciences in 1998. For the sake of clarity, I decided to deal only with manifolds without boundary. An appendix concerning manifolds with boundary can be found at the end of these notes. To illustrate some of the results or concepts developed here, I have included some discussions of a special family of PDEs where these results and concepts are used. These PDEs are generalizations of the scalar curvature equation. As is well known, geometric problems often lead to limiting cases of known problems in analysis.

The study of Sobolev spaces on Riemannian manifolds is a field currently undergoing great development. Nevertheless, several important questions still puzzle mathematicians today. While the theory of Sobolev spaces for noncompact manifolds has its origin in the 1970s with the work of Aubin, Cantor, Hoffman, and Spruck, many of the results presented in these lecture notes have been obtained in the 1980s and 1990s. This is also the case for the applications already mentioned to scalar curvature and generalized scalar curvature equations. A substantial part of these notes is devoted to the concept of best constants. This concept appeared very early on to be crucial for solving limiting cases of some partial differential equations. A striking example of this was the major role that best constants played in the Yamabe problem.

These lecture notes are intended to be as self-contained as possible. In particular, it is not assumed that the reader is familiar with differentiable manifolds and Riemannian geometry. The present notes should be accessible to a large audience, including graduate students and specialists of other fields.

The present notes are organized into nine chapters. Chapter 1 is a quick introduction to differential and Riemannian geometry. Chapter 2 deals with the general theory of Sobolev spaces for compact manifolds, while Chapter 3 deals with the general theory of Sobolev spaces for complete, noncompact manifolds. Best constants problems for compact manifolds are discussed in Chapters 4 and 5, while Chapter 6 deals with some special type of Sobolev inequalities under constraints. Best constants problems for complete noncompact manifolds are discussed in Chapter 7. Chapter 8 deals with Euclidean-type Sobolev inequalities. The influence of symmetries on Sobolev embeddings is discussed in Chapter 9. An appendix at the end of these notes briefly discusses the case of manifolds with boundaries.

It is my pleasure to thank my friend Jalal Shatah for encouraging me to write these notes. It is also my pleasure to express my deep thanks to my friends and colleagues Tobias Colding, Zindine Djadli, Olivier Druet, Antoinette Jourdain, Michel Ledoux, Frédéric Robert, and Michel Vaugon for stimulating discussions and valuable comments about the manuscript. Finally, I wish to thank Reeva Goldsmith, Paul Monsour, and Joe Shearer for the wonderful job they did in the preparation of the manuscript.

Emmanuel Hebey
Paris, September 1998

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EMMANUEL HEBEY

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