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LOUIS NIRENBERG

LECTURE NOTES

Topics in Nonlinear Functional Analysis

Notes by Ralph A. Artino

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6 Topics in Nonlinear Functional Analysis

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Contents

Preface to New Edition

These lecture notes are presented here unchanged from the 1974 edition (except that the proof of Proposition 1.7.2 has been changed).

Since 1974 many books on nonlinear functional analysis have appeared. Furthermore, variational methods in nonlinear functional analysis, which are not discussed here, have seen enormous development. Here are a few more up-to-date references:

Ambrosetti, A., and Prodi, G.: *A Primer of Nonlinear Analysis.* Cambridge Studies in Advanced Mathematics, 34. Cambridge University Press, Cambridge, 1993.

Chang, K.-C.: *Infinite-Dimensional Morse Theory and Multiple Solution Problems.* Progress in Nonlinear Differential Equations and Their Applications, 6. Birkhäuser, Boston, 1993.

Deimling, K.: *Nonlinear Functional Analysis.* Springer, Berlin–New York, 1985.

Ekeland, I.: *Convexity Methods in Hamiltonian Mechanics.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 19. Springer, Berlin, 1990.

Ghoussoub, N.: *Duality and Perturbation Methods in Critical Point Theory.* Cambridge Tracts in Mathematics, 107. Cambridge University Press, Cambridge, 1993.

Ize, J.: *Bifurcation Theory for Fredholm Operators.* Mem. Amer. Math. Soc. **7** (1976), no. 174, viii + 128 pp.

Mawhin, J.: *Topological Degree Methods in Nonlinear Boundary Value Problems.* Expository lectures from the CBMS Regional Conference held at Harvey Mudd College, Claremont, Calif., June 9–15, 1977. CBMS Regional Conference Series in Mathematics, 40. American Mathematical Society, Providence, R.I., 1979.

Mawhin, J., and Willem, M.: *Critical Point Theory and Hamiltonian Systems.* Applied Mathematical Sciences, 74. Springer, New York–Berlin, 1989.

Rabinowitz, P. H.: *Minimax Methods in Critical Point Theory with Applications to Differential Equations.* CBMS Regional Conference Series in Mathematics, 65. American Mathematical Society, Providence, R.I., 1986.

Schechter, M.: *Linking Methods in Critical Point Theory.* Birkhäuser, Boston, 1999.

Struwe, M.: *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*. Springer, Berlin, 1990.

Willem, M.: *Minimax Theorems.* Progress in Nonlinear Differential Equations and Their Applications, 24. Birkhäuser, Boston, 1996.

Zeidler, E.: *Nonlinear Functional Analysis and Its Applications.* I. *Fixed-Point Theorems*. Springer, New York–Berlin, 1986.

Zeidler, E.: *Nonlinear Functional Analysis and Its Applications.* II/B. *Nonlinear Monotone Operators*. Springer, New York–Berlin, 1990.

A Russian edition of the notes was published in 1977. It contains an extra section and a very long list of further references.

Preface

In this course we shall take up a variety of topological and analytic techniques for the study of nonlinear problems, and we shall illustrate their use by applications to nonlinear differential and integral equations, primarily to rather simple nonlinear elliptic equations.

We begin with degree of mapping, first in finite dimensions and then in Banach space—the Leray-Schauder degree theory—as well as extensions of this theory. This is used in the study of existence of global solutions of nonlinear problems and also in local, perturbation problems. Concerning the latter we shall spend considerable time on bifurcation problems, i.e., problems in which various solutions may branch from a particular one.

A few topics in the calculus of variations will be treated, such as monotone operators and min-max theorems. We will also study the deep Nash-Moser extension of the implicit function theorem.

Concerning the background for the course, students should know standard linear operator theory. We also assume familiarity with basic notions of differentiable manifolds and differential forms. Almost no knowledge of topology is assumed. Occasionally some well-known results of homotopy theory will be cited without proof.

The principal reference for the course is the book:

Schwartz, J. T.: *Nonlinear Functional Analysis*. Gordon and Breach, New York, 1969, [**11**].

We now list some references to degree theory, applications, and to bifurcation theory. For general background on degree theory:

Krasnosel'skii, M. A.: *Topological Methods in the Theory of Nonlinear Integral Equations*. Macmillan, New York, 1964, [**6**].

Milnor, J. W.: *Topology from the Differentiable Viewpoint.* University Press of Virginia, Charlottesville, Va., 1965, [**8**].

Vainberg, M. M.: *Variational Methods for the Study of Nonlinear Operators*. Holden-Day, San Francisco–London–Amsterdam, 1964, [**13**].

A number of applications of degree theory may be found in the papers of

Zarantonello, E. H. (ed.): *Contributions to Nonlinear Functional Analysis*. Academic Press, New York–London, 1971, [**4**].

For recent developments and extensions of degree theory and fixed-point theory, see:

Granas, A.: *Topics in Infinite Dimensional Topology.* Sém. Collège de France, 1969–70, [**5**].

On bifurcation theory:

Aĭzengendler, P. G., and Vaĭnberg, M. M.: Methods of investigation in the theory of branching of solutions. *Mathematical Analysis 1965 (Russian)*, 7–69. Akad. Nauk SSSR Inst. Naučn. Informacii, Moscow, 1966. Translation in 1–72, *Progress in Math.*, vol. 2. Plenum, New York, 1968, [**1**].

Keller, J. B., and Antman, S. (eds.): *Bifurcation Theory and Nonlinear Eigenvalue Problems.* Benjamin, New York–Amsterdam, 1969, [**3**].

***Rocky Mountain J. Math.*:** Spring 1973, vol. 3, no. 2, the entire issue, [**9**].

Sattinger, D. H.: *Topics in Stability and Bifurcation Theory.* Springer Lecture Notes, No. 309. Springer, Berlin–New York, 1973, [**10**].

Stakgold, I.: Branching of solutions of nonlinear equations. *SIAM Rev.* 13: 289–332, 1971, [**12**].

Vaĭnberg, M. M., and Trenogin, V. A.: The Ljapunov and Schmidt methods in the theory of non-linear equations and their subsequent development. (Russian) *Uspehi Mat. Nauk* 17(2/104): 13–75, 1962. Translation in *Russian Math. Surveys* 17: 1–60, 1962, [**14**].

Many other interesting nonlinear problems are treated in:

Berger, M., and Berger, M.: *Perspective in Nonlinearity.* Benjamin, New York–Amsterdam, 1968, [**2**].

Lions, J. L.: *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Gauthier-Villars, Paris, 1969, [**7**].

Further references are given in the notes and are collected in the bibliography at the end.

A number of people contributed greatly to the course and the notes. The latter part of the course was conducted as a seminar, and the lectures of several participants, though not all, are included here. My warm thanks, in particular, to J. A. Ize for Sections 4.4 through 4.7 and his contributions throughout the notes, and also to E. Zehnder for his generous exposition on general implicit function theorems for Chapter 6 (which he also wrote). I also wish to thank Ralph Artino for writing the notes, and John Tavantzis for catching many errors. In addition, my thanks to Connie Engle for her cheerful and excellent typing.

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Topics in Nonlinear Functional Analysis

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From reviews for the First Edition:

These lecture notes are extremely stimulating.

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[The book] is short, concise, and to the point, and the proofs are unusually elegant, always with a geometric flavor and the best available.

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Since its first appearance as a set of lecture notes published by the Courant Institute in 1974, this book served as an introduction to various subjects in nonlinear functional analysis. The current edition is a reprint of these notes, with added bibliographic references.

Topological and analytic methods are developed for treating nonlinear ordinary and partial differential equations. The first two chapters of the book introduce the notion of topological degree and develop its basic properties. These properties are used in later chapters in the discussion of bifurcation theory (the possible branching of solutions as parameters vary), including the proof of Rabinowitz's global bifurcation theorem. Stability of the branches is also studied. The book concludes with a presentation of some generalized implicit function theorems of Nash-Moser type with applications to Kolmogorov-Arnold-Moser theory and to conjugacy problems.

After more than 20 years, this book continues to be an excellent graduate level textbook and a useful supplementary course text.

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