

C O U R A N T

9

ANDREW MAJDA

LECTURE
NOTES

Introduction to PDEs
and Waves for the
Atmosphere and
Ocean

American Mathematical Society
Courant Institute of Mathematical Sciences



Introduction to PDEs and Waves for the Atmosphere and Ocean

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Contents

Preface	ix
Chapter 1. Introduction	1
1.1. Basic Properties of the Equations with Rotation and Stratification	1
1.2. Two-Dimensional Exact Solutions	2
1.3. Buoyancy and Stratification	5
1.4. Jet Flows with Rotation and Stratification	6
1.5. From Vertical Stratification to Shallow Water	6
Chapter 2. Some Remarkable Features of Stratified Flow	9
2.1. Energy Principle	9
2.2. Vorticity in Stratified Fluids and Exact Solutions Motivated by Local Analysis	11
2.3. Use of Theorem 2.4: Exact Two-Dimensional Solutions	16
2.4. Nonlinear Plane Waves in Stratified Flow: Internal Gravity Waves	20
2.5. Exact Solutions with Large-Scale Motion and Nonlinear Plane Waves	24
2.6. More Details for Theorem 2.7 on Special Exact Solutions for the Boussinesq Equations Including Plane Waves	26
Chapter 3. Linear and Nonlinear Instability of Stratified Flows with Strong Stratification	31
3.1. Boussinesq Equations and Vorticity Stream Formulation	33
3.2. Nonlinear Instability of Stratified Flows	37
3.3. Shear Flows	40
3.4. Some Background Facts on ODEs	44
Chapter 4. Rotating Shallow Water Theory	49
4.1. Rotating Shallow Water Equations	49
4.2. Conservation of Potential Vorticity	52
4.3. Nonlinear Conservation of Energy	53
4.4. Linear Theory for the Rotating Shallow Water Equations	54
4.5. Nondimensional Form of the Rotating Shallow Water Equations	60
4.6. Derivation of the Quasi-Geostrophic Equations	62
4.7. The Quasi-Geostrophic Equations as a Singular PDE Limit	65
4.8. The Model Rotating Shallow Water Equations	67
4.9. Preliminary Mathematical Considerations	69

4.10.	Rigorous Convergence of the Model Rotating Shallow Water Equations to the Quasi-Geostrophic Equations	73
4.11.	Proof of the Convergence Theorem	75
Chapter 5.	Linear and Weakly Nonlinear Theory of Dispersive Waves with Geophysical Examples	79
5.1.	Linear Wave Midlatitude Planetary Equations	79
5.2.	Dispersive Waves: General Properties	82
5.3.	Interpretation of Group Velocity	86
5.4.	Distant Propagation from a Localized Source	90
5.5.	WKB Methods for Linear Dispersive Waves	93
5.6.	Beyond Caustics: Eikonal Equation Revisited	107
5.7.	Weakly Nonlinear WKB for Perturbations Around a Constant State	109
5.8.	Nonlinear WKB and the Boussinesq Equations	117
Chapter 6.	Simplified Equations for the Dynamics of Strongly Stratified Flow	125
6.1.	Nondimensionalization of the Boussinesq Equations for Stably Stratified Flow	126
6.2.	The Vorticity Stream Formulation and Elementary Properties of the Limit Equations for Strongly Stratified Flow	133
6.3.	Solutions of the Limit Dynamics with Strong Stratification as Models for Laboratory Experiments	136
Chapter 7.	The Stratified Quasi-Geostrophic Equations as a Singular Limit of the Rotating Boussinesq Equations	147
7.1.	Introduction	147
7.2.	The Rotating Boussinesq Equations	148
7.3.	The Nondimensional Rotating Boussinesq Equations	151
7.4.	Formal Asymptotic Derivation of the Quasi-Geostrophic Equations as a Distinguished Asymptotic Limit of Small Rossby and Froude Numbers	153
7.5.	Rigorous Convergence of the Rotating Boussinesq Equations to the Quasi-Geostrophic Equations	157
7.6.	Preliminary Mathematical Considerations	160
7.7.	Proof of the Convergence Theorem	164
Chapter 8.	Introduction to Averaging over Fast Waves for Geophysical Flows	171
8.1.	Introduction	171
8.2.	Motivation for Fast-Wave Averaging	171
8.3.	A General Framework for Averaging over Fast Waves	173
8.4.	Elementary Analytic Models for Comparing Instabilities at Low Froude Numbers with the Low Froude Number Limit Dynamics	177
8.5.	The Rapidly Rotating Shallow Water Equations with Unbalanced Initial Data in the Quasi-Geostrophic Limit	182

8.6. The Interaction of Fast Waves and Slow Dynamics in the Rotating Stratified Boussinesq Equations	191
Chapter 9. Waves and PDEs for the Equatorial Atmosphere and Ocean	199
9.1. Introduction to Equatorial Waves for Rotating Shallow Water	199
9.2. The Equatorial Primitive Equations	207
9.3. The Nonlinear Equatorial Long-Wave Equations	220
9.4. A Simple Model for the Steady Circulation of the Equatorial Atmosphere	226
Bibliography	233

Preface

These lecture notes are based on material presented by the author in graduate courses at the Courant Institute in 1995, 1997, 1999, and 2001. The lectures emphasize the serendipity between modern applied mathematics and geophysical flows in the style of modern applied mathematics where rigorous analysis, as well as asymptotic, qualitative, and numerical modeling all interact. The goal was to introduce mathematicians to atmosphere/ocean science (AOS) in this fashion and conversely to develop a set of lecture notes of interest to the disciplinary community ranging from graduate students to researchers in AOS. During these courses, the beautiful applications-oriented text by Adrian Gill [11] and the well-known text by Pedlosky [29] were often used for supplementary reading material outside class. These lecture notes do not require a previous course in fluid dynamics although the texts [2, 19] are recommended for supplementary material on incompressible flow while the texts [4, 26, 33] are suggested for additional interesting topics in the mathematical physics of geophysical flows.

The author thanks Prof. Pedro Embid and his former Ph.D. student Jonathan Callet for their help with early versions of the lecture notes for Chapters 2, 4, 5, and 7. Some joint research with Professor Embid that was developed as an outgrowth of the earliest lecture courses as well as joint work with my former Courant postdocs Prof. Marcus Grote and Misha Shefter has been incorporated into the material presented here; their explicit and implicit contributions are acknowledged warmly. My current Courant postdoc, Boualem Khouider, has been a great help with the notes for Chapter 9, which were developed as part of a lecture course in spring 2001. Finally, the author acknowledges the generous support of both the National Science Foundation and the Office of Naval Research during the development of these lecture notes, including partial salary support for Professor Embid during his visit to the Courant Institute during the mid 1990s.

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Introduction to PDEs and Waves for the Atmosphere and Ocean

ANDREW MAJDA

The goals of these lecture notes, based on courses presented by the author at the Courant Institute of Mathematical Sciences, are to introduce mathematicians to the fascinating and important area of atmosphere/ocean science (AOS) and, conversely, to develop a mathematical viewpoint on basic topics in AOS of interest to the disciplinary AOS community, ranging from graduate students to researchers. The lecture notes emphasize the serendipitous connections between applied mathematics and geophysical flows in the style of modern applied mathematics, where rigorous mathematical analysis as well as asymptotic, qualitative, and numerical modeling all interact to ease the understanding of physical phenomena. Reading these lecture notes does not require a previous course in fluid dynamics, although a serious reader should supplement these notes with material such as additional information on geophysical flows, as suggested in the preface.

The book is intended for graduate students and researchers working in interdisciplinary areas between mathematics and atmosphere/ocean science.



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