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PETER D. LAX

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Hyperbolic Partial Differential Equations

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Hyperbolic Partial Differential Equations

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With an Appendix by Cathleen S. Morawetz

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Foreword

The theory of hyperbolic equations is a large subject, and its applications are many: fluid dynamics and aerodynamics, the theory of elasticity, optics, electromagnetic waves, direct and inverse scattering, and the general theory of relativity.

The first seven chapters of this book, based on notes of lectures delivered at Stanford in the spring and summer of 1963, deal with basic theory: the relation of hyperbolicity to the finite propagation of signals, the concept and role of characteristic surfaces and rays, energy, and energy inequalities.

The structure of solutions of equations with constant coefficients is explored with the help of the Fourier and Radon transforms. The existence of solutions of equations with variable coefficients with prescribed initial values is proved using energy inequalities. The propagation of singularities is studied with the help of progressing waves.

Chapter 8 of the second part describes finite difference approximations of hyperbolic equations. This subject is obviously of great importance for applications, but also intriguing for the theorist. The proof of stability of difference schemes is analogous to the derivation of energy estimates, but much more sophisticated.

Chapter 9 presents a streamlined version of the Lax-Phillips scattering theory. The last section describes the Pavlov-Faddeev analysis of automorphic waves, and their mysterious connection to the Riemann hypothesis.

Chapter 10, the only one dealing with nonlinear waves, is about hyperbolic systems of conservation laws, an active research area today. We present the basic concepts and results.

Five brief appendices sketch topics that are important or amusing, such as Huygens' principle, a theory of mixed initial and boundary value problems, and the use of nonstandard energy identities.

I hope that this book will serve well as an introduction to the multifaceted subject of hyperbolic equations.

Peter Lax
New York
February 2006

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Hyperbolic Partial Differential Equations

PETER D. LAX

The theory of hyperbolic equations is a large subject, and its applications are many: fluid dynamics and aerodynamics, the theory of elasticity, optics, electromagnetic waves, direct and inverse scattering, and the general theory of relativity. This book is an introduction to most facets of the theory and is an ideal text for a second-year graduate course on the subject.

The first part deals with the basic theory: the relation of hyperbolicity to the finite propagation of signals, the concept and role of characteristic surfaces and rays, energy, and energy inequalities. The structure of solutions of equations with constant coefficients is explored with the help of the Fourier and Radon transforms. The existence of solutions of equations with variable coefficients with prescribed initial values is proved using energy inequalities. The propagation of singularities is studied with the help of progressing waves.

The second part describes finite difference approximations of hyperbolic equations, presents a streamlined version of the Lax-Phillips scattering theory, and covers basic concepts and results for hyperbolic systems of conservation laws, an active research area today.

Four brief appendices sketch topics that are important or amusing, such as Huygens' principle and a theory of mixed initial and boundary value problems. A fifth appendix by Cathleen Morawetz describes a nonstandard energy identity and its uses.

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