Stochastic Processes
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16 Stochastic Processes

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Dedication

To Gopal

I had planned to complete this book within a short time of the publication of the volume on probability theory. But the events of September 11, 2001, intervened. We lost our son Gopal that day, a victim of violence in the name of God. I dedicate this volume to his memory.
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Preface

This is a continuation of the volume on probability theory and likewise covers the contents of courses given at the Courant Institute. This volume deals with certain elementary continuous-time processes. We start with a description of the Poisson process and related processes with independent increments. After a brief look at Markov processes with a finite number of jumps we proceed to study Brownian motion. We then go on to develop stochastic integrals and Itô’s theory in the context of one-dimensional diffusion processes. It ends with a brief survey of the general theory of Markov processes.
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Stochastic Processes
S. R. S. VARADHAN

This is a brief introduction to stochastic processes studying certain elementary continuous-time processes. After a description of the Poisson process and related processes with independent increments as well as a brief look at Markov processes with a finite number of jumps, the author proceeds to introduce Brownian motion and to develop stochastic integrals and Itô's theory in the context of one-dimensional diffusion processes. The book ends with a brief survey of the general theory of Markov processes.

The book is based on courses given by the author at the Courant Institute and can be used as a sequel to the author's successful book Probability Theory in this series.

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