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Random Matrix Theory: Invariant Ensembles and Universality

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Dedication

To Rebecca and Abby — *P.D.*

To my wife Diana Katsman and my mother Natalia Barkova — *D.G.*

Contents

Preface

This book is based in part on a graduate course given by the first author at the Courant Institute in fall 2005. Subsequently, the second author gave a modified version of this course at the University of Rochester in spring 2007. In an earlier book on the subject [**21**] the author considered only unitary ensembles; here the primary focus is on orthogonal and symplectic ensembles.

In the first part of this book we present a unified treatment of the algebraic aspects of the unitary, orthogonal, and symplectic ensembles, following the approach of Tracy and Widom [**99**] and Widom [**103**]. The second part of the book contains an exposition of the work of the authors on the proof of universality in the bulk for orthogonal and symplectic ensembles in [**24**]. A proof of universality in the bulk for unitary ensembles can be found in [**21**].

Universality at the edge for the three types of invariant ensembles was addressed in [**23**]. The proof of universality in the bulk and at the soft and hard spectral edges for orthogonal and symplectic ensembles with generalized Laguerre-type weights using the methods of [**23, 24**] was given in [**25**]. In this introductory text, however, we will limit our presentation to results from [**24**]. We will, however, incorporate the streamlined proof of an important technical result (see Section 6.8) from [**16**]: this replaces the more cumbersome approach in [**23, 24**].

We note that in this book we prove quantitative versions of the error estimates for the Widom correction terms for orthogonal and symplectic ensembles with generalized Hermite-type weights. This is in contrast to [**23, 24**], where the authors prove only $o(1)$ estimates for the errors (see Section 1.3).

The first author was supported in part by NSF Grant DMS–0500923. The second author was supported in part by NSF Grant DMS-0556049; he would like to thank the University of Rochester for the opportunity to serve on the faculty and for the very conducive working atmosphere. In addition, he greatly appreciates the value that Wilshire Associates Incorporated places on basic research, and is grateful for the arrangement that allowed him to complete the work on the present monograph.

The authors would also like to thank Alexei Borodin, Thomas Kriecherbauer, and Chris Sinclair for very useful comments and information. Finally, the authors would like to thank Paul Monsour for his masterful editing of the manuscript, for his patience, and for his good cheer in the face of many last-minute changes to the text.

Percy Deift and Dimitri Gioev

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Random Matrix Theory: Invariant Ensembles and Universality

PERCY DEIFT AND DIMITRI GIOEV

This book features a unified derivation of the mathematical theory of the three classical types of invariant random matrix ensembles—orthogonal, unitary, and symplectic. The authors follow the approach of Tracy and Widom, but the exposition here contains a substantial amount of additional material, in particular, facts from functional analysis and the theory of Pfaffians. The main result in the book is a proof of universality for orthogonal and symplectic ensembles corresponding to generalized Gaussian type weights following the authors' prior work. New, quantitative error estimates are derived.

The book is based in part on a graduate course given by the first author at the Courant Institute in fall 2005. Subsequently, the second author gave a modified version of this course at the University of Rochester in spring 2007. Anyone with some background in complex analysis, probability theory, and linear algebra and an interest in the mathematical foundations of random matrix theory will benefit from studying this valuable reference.

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