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LECTURE NOTES

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An Introduction to Theoretical Fluid Mechanics

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Contents

Preface

These notes were prepared for a one-semester graduate course in introductory classical fluid mechanics. The fluid mechanics curriculum at the Courant Institute has traditionally consisted of a two-semester introductory sequence, followed by special topic courses. It was common to treat incompressible fluids, both ideal and viscous, in the first semester, and then to move to compressible flow, gas dynamics, and shock waves in the second. Because of the pressures of time and course scheduling, and the ever-expanding scope of the subject matter, a decision was made to offer instead a one-semester introductory course, which would include at least some of the material on compressible flow, to be followed by a second-semester special topic fluids course that could change from year to year depending upon faculty and student interests.

The present course was developed for students with a strong undergraduate mathematics background, but I have assumed no previous exposure to fluid mechanics. The selected material is fairly standard, but it was chosen to emphasize the mathematical methods that have their origin in fluid theory. A central problem of the classical theory is the subtle relation between an ideal fluid and a real fluid of small viscosity (or more precisely, a fluid flow with a large Reynolds number). Many of the crowning achievements of the fluid dynamicists of the nineteenth and twentieth centuries, certainly including Prandtl's boundary layer theory, airfoil theory, much of the theory of singular perturbations, and the recent developments surrounding triple-deck theory, are all motivated by this problem. I have tried to keep some of these issues front and center when presenting the classical results in potential flow and in models for the lift and drag of bodies in a flow. Some attention is paid to the problem of locomotion in fluids, since it provides an interesting example where both Eulerian and Lagrangian methods play a role.

As a course in a mathematics curriculum, fluid mechanics should, in my opinion, be presented as a beautiful, practical subject, involving a moving continuum whose deformations are determined by certain natural physical laws. But the mathematical complexity of the subject is legion; to take one of many examples, the global existence of solutions of the Navier-Stokes equations for an incompressible fluid remains an open question. In an introductory course we must be content with the relatively small number of model problems that convey the flavor of the subject without excessive analysis.

The choices made here leave out, or only touch upon, many interesting and important topics. Among these are turbulence, shallow-water theory, rotating fluids and associated geophysical models, water waves, hydrodynamic stability, surface tension phenomena, and, importantly, computational fluid dynamics. Nevertheless, it is hoped that these notes offer a fair introduction to the classical theory and a preparation for more specialized courses in fluid mechanics.

STEPHEN CHILDRESS

Supplementary Notes

Chapters 1 and 2. Much of the background material in these notes is covered in the excellent classical texts [**1, 5, 8**]. Our discussion of compressible flow is based for the most part on [**2**]. The other selected reference texts above contain useful supplementary material. In particular [**7**] begins with a thorough treatment of fluid kinematics. The classic work of L. Prandtl [**9**] is a rich source of ideas basic to modern fluid dynamics. Also to be recommended is the review by James Serrin, "Mathematical principles of classical fluid mechanics," *Handbuch der Physik*, vol. VIII/I, pp. 125–263, Springer, 1959.

While the basic laws of mechanics that underlie our subject are due to Newton (1642–1727), the mathematical formulation of classical fluid dynamics was largely the creation of J. L. d'Alembert (1717–1783), J. and D. Bernoulli (1667–1748 and 1700–1782), and especially L. Euler (1707–1783). An early description of the "Eulerian" viewpoint was put forward in 1749 by d'Alembert, but as Lamb [**4**] notes, Euler generalized this approach and also introduced the "Lagrangian" form of fluid kinematics. Two of three fundamental papers in hydrodynamics, written by Euler in 1755 and published in 1757, are translated in the recent volume celebrating the 250th anniversary of Euler's work (see [**10**]). The reader interested in the history of fluid dynamics will want to consult this valuable reference. In particular, the paper "From Newton's mechanics to Euler's equations" by O. Darrigol and U. Frisch in [**10**] provides an excellent discussion of the emergence of crucial ideas such as that of fluid pressure and the convective component of acceleration from the dynamical models of fluid motion prevalent in that day.

Chapter 3. We refer to the monograph of Truesdell [**21**] for a discussion of the history and evolution of the theory of vorticity and circulation. For an excellent survey of many problems of vortex dynamics, see the monograph of P. G. Saffman [**19**]. The work of Hermann von Helmholtz (1821–1894) and W. Thomson (Lord Kelvin, 1824–1907) laid out the basic laws of vortex theory, although the vector field $\nabla \times \mathbf{u}$ appears in d'Alembert's work. However, d'Alembert wrongly assumed that steady flows of an ideal incompressible fluid were necessarily irrotational.

An important outstanding problem of classical fluid mechanics concerns the evolution of the vorticity field in the initial value problem for Euler flow of an ideal incompressible fluid when the initial flow field is smooth. What is the largest ultimate rate of growth in time of the magnitude of the vorticity? How fast does the $\int \omega^2 \, dV$ ultimately grow with time? Is it possible that vorticity could become infinite somewhere in a finite time? The question is discussed in papers in [**10**], where many references to recent work may be found.

Chapter 4. Paul R. H. Blasius (1883–1970) was one of the first students of Prandtl. The famous Blasius formulas appeared in 1910 (*Z. Math. Phys.* **58**, pp. 90–110). Locomotion in an ideal fluid was first studied by P. G. Saffman in 1968 (*J. Fluid Mech.* **28**, pp. 385–389). Calculations of the virtual mass of bodies is discussed in most texts; see especially [**1, 5, 8**]. Darwin's theorem was the work of Charles Galton Darwin (1887–1962), a physicist and grandson of Charles Darwin. He derived the theorem in 1953 (*Proc. Camb. Phil. Soc.* **49**, pp. 342–354).

Chapter 5. Calculations of drag lift of bodies moving in fluids have been central to our subject since the time of Newton. D'Alembert's finding of zero drag in potential flow accounted for the early view of theoretical fluid dynamics as unphysical and predictive of absurd conclusions. The reconciliation of d'Alembert's paradox with the modern theory of viscous fluids is a continuing theme in fluid mechanics; see the excellent review by K. Stewartson, "D'Alembert's paradox," *SIAM Review*, vol. 23, no. 3, pp. 308–343, 1981.

It is fair to say that the Kutta-Joukowski theory is incomplete in a larger sense of providing a theory of the inviscid limit. The choice of the proper selection rules for inviscid flows, accounting under various conditions for the effects of small viscosity, remains an unsolved problem of fluid dynamics. Vol. III of [**3**] contains a section by Witoszyński and Thompson on "The theory of single burbling," a largely forgotten attempt to improve on the lift computed in K-J theory by a modification of conditions at the trailing edge. Recent efforts have tried to generalize the K-J condition to unsteady flow; see D. G. Crighton, *Ann. Rev. Fluid Mech.* **17**, pp. 411–445, 1985, and also Allan D. Pierce, *J. Acoust. Soc. Am.* **109**, no. 5, pp. 2469–2470, 2001.

For a useful collection of articles covering the aerodynamic theory of lift and drag used in the design of early aircraft and many details concerning airfoil design, see [**3**]. In particular, we remark that the cusped trailing edge of the Joukowski foils is not well suited to wing fabrication, and foils having a finite tangent angle at the trailing edge are desirable. These are provided by the *Kármán-Trefftz* family. Writing our mapping $z = Z + a^2/Z$ in the form

$$\frac{z+2a}{z-2a} = \left(\frac{Z+a}{Z-a}\right)^2,$$

the *Kármán-Trefftz* theory generalizes this to

$$\frac{z+na}{z-na} = \left(\frac{Z+a}{Z-a}\right)^n.$$

If $n = 2 - \frac{\gamma}{\pi}$, it can be shown that the tangents at the trailing edge form an angle γ.

For an extended discussion of wakes and drag, including experimental results, see [**14**].

Many additional examples of free-streamline flow calculations can be found in [**8**]. A discussion of Prandtl's lifting-line theory as a singular perturbation problem for large aspect ratio is described in [**22**].

Erich Trefftz (1888–1937) was a German mathematician and aerodynamicist. He studied at Aachen and then at Göttingen, where he was a student of Hilbert and

Prandtl. His research included key problems of hydrodynamics, but ranged widely in applied and numerical mathematics.

Chapters 6 and 7. The concept of stress is due to A. L. Cauchy (1789–1857) from a paper in 1823, although the physics of fluid friction goes back to Newton. Claude-Louis Navier (1785–1836) and George G. Stokes (1819–1903) developed the equations we study here for Newtonian viscous flow. Jean Louis Marie Poiseuille (1797–1869) was a French physician and physiologist whose experiments in 1840 studied viscous flow through long, thin tubes. J. M. Burgers (1895–1981) was a Dutch physicist whose research included many fundamental aspects of fluid dynamics. A variety of examples of viscous flow may be found in the review by R. Berker, "Intégration des équations du mouvement d'un fluide visqueux incompressible," *Handbuch der Physik*, vol. VIII/II, Springer (1963).

For a very complete text on creeping flows with emphasis on flows past bodies, see J. Happel and H. Brenner, *Low Reynolds number hydrodynamics*, 2nd ed., Springer (1983). Stokes noted in 1851 the nonexistence of a solution of his equations appropriate to a circular cylinder in a uniform infinite flow. Carol Wilhelm Oseen (1879–1944) introduced his equations and succeeded in 1910 in showing that they allowed a solution of this problem for small Reynolds numbers. A discussion of the solution may be found in [**4**]. Oseen went on to publish his famous monograph *Neue Methoden und Ergebnisse in der Hydrodynamik*, Akademische Verlagsgesellschaft, Leipzig, in 1927, where many problems involving fluid inertia are worked out in the linear setting provided by his equations. The modern analysis of Stokes' paradox is based on the seminal paper of S. Kaplun, "Low Reynolds number flow past a circular cylinder," *J. Math. Mech.* **6**, pp. 595–603, 1957; see also [**22**]. In this work the Stokes equations emerge from an inner limit and the Oseen equations from an outer limit. A formal procedure of matching of the two sets of solutions completely resolves the singular nature of the low Reynolds number limit in the neighborhood of infinity.

For a discussion of locomotion in Stokes flow, see the delightful paper of E. M. Purcell, "Life at low Reynolds number," *Am. J. Phys.* **45**, pp. 3–11, 1977, and also M. J. Lighthill, *Mathematical Biofluiddynamics*, SIAM (1987), and S. Childress, *Mechanics of Swimming and Flying*, Cambridge University Press (1981). The scallop theorem applies to a single locomoting body, but not to multiple interacting Stokesian swimmers; see E. Lauga and D. Bartolo, "No many-scallop theorem: Collective locomotion of reciprocal swimmers," *Phys. Rev. E* **78**, 030901(R) (2008). Also, the constraint posed by the scallop theorem is broken at finite Reynolds number, however small, but can lead for certain symmetric movements to a bifurcation to locomotion at a positive critical Reynolds number; see S. Childress et al., "Symmetry breaking leads to forward flapping flight," *J. Fluid Mech.* **506**, pp. 147–155 (2004).

Chapter 8. For a discussion of early examples of the boundary layer concept, see M. Van Dyke, "Nineteenth-century roots of the boundary-layer idea," *Siam Review* **36**, pp. 415–424, 1994. For discussion of a range of topics in classical boundary layer theory, see [**14**] and especially H. Schlichting and K. Gersten, *Boundary-layer theory*, 8th ed., Springer (2000). The numerical solution of the Prandtl equations was given by Blasius in his 1907 dissertation; see *Zeitschr. Math. Phys.* **56**, pp. 1–37, 1908. The solutions of Falkner and Skan appeared in *Philos. Mag.* **12**, pp. 865–896, 1931. For an interesting alternative treatment of the Blasius and Falkner-Skan problems, see H. Weyl, "On the differential equations of the simplest boundary-layer problems," *Ann. of Math. (2)* **43**, pp. 381–407, 1942. The one-dimensional boundary layer model of Section 8.3 appeared in *Fluid dynamics*, K. O. Friedrichs and R. von Mises, Springer (1971). The modern treatment of fluid boundary layer theory as a singular perturbation problem is discussed in [**22**]; see also [**16**]. A landmark paper by S. Kaplun, "The role of coordinate systems in boundary-layer theory," *Z. Angew. Math. Physik* **5**, pp. 111–135, 1954, showed that there exist optimal coordinate systems for capturing the flow due to displacement thickness within the boundary layer limit. The triple-deck theory emerged from the work of many researchers, including Goldstein, Kaplun, Messiter, Neiland, Smith, Stewartson, and Sychev. For reviews of the triple-deck theory of laminar separation, see F. T. Smith, "On the high Reynolds number theory of laminar flows," *IMA J. Appl. Math* **28**, pp. 207–281, 1982, and R. E. Meyer, "A view of the triple deck," *SIAM J. Appl. Math.* **43**, pp. 639–663, 1983. For a survey of rigorous mathematical results for the Prandtl equations, see K. Nickel, "Prandtl's boundary-layer theory from the viewpoint of a mathematician," *Ann. Rev. Fluid Mech.* **5**, pp. 405–428, 1973. For the origins of Prandtl-Batchelor theory, see L. Prandtl, "Über Flüssigkeitsbewegung bei sehr kleiner Reibung," *Gesammelte Abhandlungen II*, pp. 575–584, Springer (1961), and G. K. Batchelor, "A proposal concerning laminar wakes behind bluff bodies at large Reynolds number," *J. Fluid Mech.* **1**, pp. 388–398, 1956.

Chapters 9–12. The general solution of the initial value problem for the wave equation in three dimensions was given by S. D. Poisson (1781–1840) in 1820 and was obtained in the form given here by G. Kirchhoff (1824–1887) in 1876; see Rayleigh's *The theory of sound*, vol. II, Dover, p. 97 (1945). Sir James Lighthill (1924–1998) and Theodore von Kármán (1881–1963) are two giants of twentieth-century fluid mechanics who contributed substantially to the development of the theory of compressible flow. The books of Lighthill [**17**] and Whitham [**25**] are excellent sources for the theory of linear and nonlinear waves in compressible fluids. In connection with viscous and weak nonlinear effects on sound waves, we mention the important review by Lighthill, "Viscosity effects in sound waves of finite amplitude," in *Surveys in mechanics*, pp. 250–351, Cambridge University Press (1956). The theory of aerodynamic generation of sound, introduced in Problem 10.7, was developed by Lighthill in the 1950s. The theory was subsequently developed and expanded by J. E. Ffowcs Williams and others, and an excellent survey is given by Ffowcs Williams in "Hydrodynamic noise," *Annu. Rev. Fluid Mech.* **1**, pp. 197–222, 1969, and "Aeroacoustics," *Annu. Rev. Fluid Mech.* **9**, pp. 447–468, 1977. For a detailed account of linearized compressible flow, see [**24**]. For a discussion of transonic similitude, see W. D. Hayes, "Pseudotransonic similitude and first-order wave structure," *J. Aero. Sci.* **21**, pp. 721–730, 1954. Hayes (1919–2001) was a student of von Kármán at CalTech. His 1947 dissertation, "Linearized supersonic flow" (see also his "Linearized supersonic flows with axial symmetry," *Quart. Appl. Math.* **4**, pp. 255–261, 1946), contained the theoretical basis for the area rule, later used in the design of transonic aircraft. Hayes's monograph [**15**] contains a full analysis of the Hugoniot curve under various physical assumptions. For discussion of the water waves, and in particular a nice treatment of shallow-water theory and its applications, see the book by J. J. Stoker [**20**].

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Index

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STEPHEN CHILDRESS

This book gives an overview of classical topics in fluid dynamics, focusing on the kinematics and dynamics of incompressible inviscid and Newtonian viscous fluids, but also including some material on compressible flow. The topics are chosen to illustrate the mathematical methods of classical fluid dynamics. The book is intended to prepare the reader for more advanced topics of current research interest.

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