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Linear Algebra II

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Linear Algebra II

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Preface

The previous volume, Linear Algebra I (LA I), provided a self-contained account of basic topics that might be covered in one semester at the introductory graduate level, ending with:

- In Chapter 5, a general discussion of diagonalization including: norm convergence of linear operators, matrix-valued power series, and use of the exponential map $\text{Exp}(A) = e^A$ to solve systems of differential equations $dX/dt = A \cdot X(t)$ in which A is diagonalizable.
- In Chapter 6 we discuss orthogonal diagonalization of operators on inner product spaces, including spectral decomposition, spectral mapping theorem, polar decomposition $T = U \cdot |T|$, and singular value decomposition.

Linear Algebra II begins with more challenging basic topics, presented in Chapters 1–3, whose contents are described below. The final Chapters 4 and 5 are different. They are intended as more-or-less independent and self-contained surveys of two special topics:

- Chapter 4: Tensor Fields, Differentiable Manifolds and Vector Calculus
- Chapter 5: Matrix Lie Groups

Both are vast subjects, so presentations in these chapters will not be as fully developed as those in preceding chapters or LA I. Our intent was to present some advanced topics that illustrate uses of linear algebra in realms beyond a standard second course in linear algebra. In practice, every instructor could choose which to present, according to his or her interests and those of the class. Each chapter in LA II begins with a detailed overview of the topics that will be covered. Here is brief description of the chapter contents.

CHAPTER 1. GENERALIZED EIGENSPACES AND THE JORDAN DECOMPOSITION. The first serious obstacle to diagonalization is dealing with nilpotent operators, which is addressed by working out a detailed procedure for computing their cyclic subspace decompositions. Although there are elegant existence proofs regarding such decompositions, the desired cyclic subspaces are not unique (unlike eigenspaces in diagonalization), and any procedure for finding them is inevitably complicated by the need to make some arbitrary choices.

For any linear operator $T : V \to V$ the space V is uniquely a direct sum (the Fitting decomposition) $V = K_\infty(T) \oplus R_\infty(T)$ of T-invariant subspaces, the "stable kernel" and "stable range" of T, on which $T|_{K_\infty}$ is nilpotent and $T|_{R_\infty}$ is bijective. If λ is an eigenvalue of T, $(T - \lambda I)$ is obviously nilpotent on its stable kernel $M_\lambda(T) = K_\infty(T - \lambda I)$, the *generalized λ-eigenspace* of T. These spaces are T-invariant and linearly independent. Whenever the characteristic polynomial $p_T(x) = \det(T - \lambda I)$ splits in $\mathbb{K}[x]$, a proof by induction on $\dim(V)$ employs the Fitting decomposition to show that V is the direct sum $V = \bigoplus_\lambda M_\lambda(T)$ of its generalized eigenspaces. Since each component $M_\lambda(T)$ has its own cyclic subspace decomposition, this "generalized eigenspace decomposition" leads directly to the Jordan canonical form $J(T)$ for T, which has many applications.

Chapter 1 ends with a brief appendix reviewing key theorems about diagonalization that were covered in Linear Algebra I, illustrating a few techniques of proof that will be relevant in discussing what to do when diagonalization fails.

CHAPTER 2. FURTHER APPLICATIONS OF THE JORDAN FORM. In the first half of this chapter we employ the Jordan form to solve higher-order ODEs by converting them into linear systems $dX/dt = A \cdot X(t)$ of constant coefficient first-order equations. We then recast the coefficent matrix A in Jordan canonical form $J(A)$ made up of diagonal blocks $B_k = \lambda_k I + E_k$ in which E_k is an elementary nilpotent matrix. Owing to nilpotence, the one-parameter groups e^{tB_k} can be written explicitly, and the system solved by taking $X(t) = e^{tJ(A)} \cdot X(0)$. The solution of the original n^{th}-order ODE is easily read from this.

These techniques work when the characteristic polynomial $p_A(x)$ splits into linear factors in the space of polynomials $\mathbb{K}[x]$. The second half of Chapter 2 concerns itself with complexifying real vector spaces and linear operators $T : V \to V$ to get complex spaces $V_\mathbb{C}$ and operators $T_\mathbb{C}$ to which the preceding methods apply. By reverse-engineering the construction of the Jordan form $J(T_\mathbb{C})$, the original $\mathbb{R}$-linear operator T can be recast in a "real-normal form" that reveals its structure. There are actually two such real forms, depending on whether $T_\mathbb{C}$ is diagonalizable or not (in which case one applies the Jordan form).

CHAPTER 3. BILINEAR, QUADRATIC, AND MULTILINEAR FORMS. Bilinear forms $B(x, y)$ on a vector space V arise often in many areas of mathematics, inner products on real vector spaces being just one example. We restrict attention to the *symmetric* and *antisymetric* B, the ones most frequently encountered. Bilinear forms can be described by $n \times n$ matrices $[B]_\mathfrak{X} = [B(e_i, e_j)]$ once a basis $\mathfrak{X} = \{e_i\}$ in V is specified. Modulo a change of basis, bilinear forms over $\mathbb{R}$ or $\mathbb{C}$ can be classified according to their normal forms, for which there are only three possibilites. We also work out the transformation law $[B]_\mathfrak{X} \to [B]_\mathfrak{Y}$ between coordinate descriptions of B.

Every bilinear form $B : V \times V \to \mathbb{K}$ determines a group $\text{Aut}(B)$ of *automorphisms*, linear operators that leave B invariant, so $B(Tx, Ty) = B(x, y)$. As examples we have the *orthogonal group* of linear rigid motions $\text{O}(n)$ on $\mathbb{R}^n$, and the *unitary operators* $\text{U}(n)$ on $\mathbb{C}^n$, that preserve the usual inner products on these spaces; there is also a third family, the *symplectic groups* $\text{Sp}(2n, \mathbb{K})$. These *classical groups* are prevalent in physics and geometry, or wherever one has to deal with symmetries. As an example we discuss the *Lorentz group*, the automorphism group $\text{Aut}(B) = \text{SO}(3,1)$ for the form

$$B(\mathbf{x},\mathbf{y}) = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4 \quad \text{for } \mathbf{x}.\mathbf{y} \in \mathbb{R}^4.$$

This is the symmetry group underlying Einstein's theory of special relativity (taking x_1, x_2, x_3 to be space coordinates and x_4 = time). It tells us how to transform measurements made in one space-time coordinate frame to those made in a different frame moving at constant velocity. (The outcome is decidedly nonintuitive!)

Beyond all this, Chapter 3 makes an initial foray into the world of *tensors* and *multilinear forms*, which have become central topics in differential geometry, and physics too — general relativity is all about tensors, as is much of electromagnetic theory and the analysis of stress in solid media. This part of Chapter 3 covers a few basic facts that set the stage for a much more extensive discussion of tensor algebra and differential geometry in Chapter 4, should the instructor choose to pursue this special topic upon completion of Chapters 1–3.

CHAPTER 4. TENSOR FIELDS, MANIFOLDS, AND VECTOR CALCULUS. This chapter explores aspects of linear and multilinear algebra that lie at the heart of modern differential geometry and ends with a reinterpretation of the main results of traditional multivariate calculus. After a brief review of the classical vector operators *div*, *grad*, and *curl* of calculus, attention shifts to the concept of a differentiable manifold and the present-day interpretation of the vector fields that live on them, which are now recognized to be of many types — fields of tangent vectors, cotangent vectors, and tensors of various ranks.

Early on, manifolds were viewed as smooth hypersurfaces embedded in some Euclidean space; tangent vectors, normal vectors, and tensor were realized as objects within that external space. But everyone knew this universal Euclidean space was a fiction. Furthermore, in trying to do calculus on, say, a 4-dimensional hypersurface M embedded in a 10-dimensional space, one is confronted with some vexing questions, for instance:

> *What do we mean by a "tangent vector to M" if we are forbidden to speak of anything external to M?*

More generally, how can one get rid of all reference to some mythical "all-encompassing Euclidean space" and deal with such questions in an intrinsic and coordinate-independent way?

A satisfactory answer emerged in the first quarter of the twentieth century, based on the concept of a *differentiable manifold* which circumvented these philosophical issues. Along with it came the realization that many different types of vector fields could be associated with a manifold. We illustrate the possibilities by examining a few examples from the physical sciences, in which we explain why:

- *velocity fields*, as in fluid flow, must be regarded as fields of tangent vectors,

- *force fields* such as electric fields, gravitational fields, etc., must be interpreted as fields of cotangent vectors.
- On the other hand, *magnetic fields* are usually described in introductory courses as vector fields, but the vectors involved are something quite different, neither tangent nor cotangent vectors. They are in fact represented by fields of rank-2 antisymmetric tensors.

Our goals in Chapter 4 are to:

- Develop enough of the theory of differentiable manifolds, and the multilinear forms (tensor fields) that live on them, to reformulate traditional multivariate calculus in modern terms.
- Work through basic concepts of multilinear algebra needed to understand the concepts upon which modern differential geometry is based.
- Provide enough exposure to these concepts so that one can carry out meaningful computations.

The last section of this chapter is focused on tying the concepts discussed in this chapter, to what you might have been taught in lower-level multivariate calculus.

CHAPTER 5. MATRIX LIE GROUPS. The subject of *Lie groups* is the nexus where three major areas of mathematics come into play — calculus-style analysis, modern algebra, and differential geometry are all intertwined in this field, underpinned by the tools linear algebra provides. That interplay among so many disciplines can be daunting, but it is also what makes Lie groups such an interesting subject.

This brief chapter cannot be a treatise on the subject. It is intended to be an introduction to the players involved, a few of the most important concepts, and examples illustrating the techniques involved in working with them. Some knowledge of differential geometry will be assumed, but the necessary topics reprised early in the chapter to make the discussion self-contained. For additional information see Section 4.1.

Many Lie groups of interest in physics and geometry were originally modeled as curvilinear smooth hypersurfaces M embedded in Euclidean spaces $\mathbb{R}^n$ (and sometimes $\mathbb{C}^n$), for example the "classical matrix groups" — the orthogonal groups $\mathrm{O}(n)$, unitary groups $\mathrm{U}(n)$, and symplectic groups $\mathrm{Sp}(2n)$ introduced in Chapter 3 as the symmetry groups associated with various bilinear forms that arise throughout mathematics — are realized as subsets of matrix space $\mathrm{M}(n, \mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Beginning in the twentieth century Lie theory (and most of differential geometry as well) moved away from the notion of embedded hypersurfaces and developed an intrinsic theory of such spaces — now called *differentiable manifolds* — in which Lie groups and manifolds were regarded as self-contained universes in their own right. However, to keep things simple we focus on *matrix Lie groups*, which can be modeled as subsets of matrix space over $\mathbb{R}$ or $\mathbb{C}$. Besides, almost everything said for matrix Lie groups carries over to the general theory of Lie groups, although some proofs get harder in that setting.

Main topics of Chapter 5 are:

- To show how differentiable structure can be imposed on a matrix group G using the implicit function theorem to define Euclidean coordinates on a family of open sets U_α that cover G. Equipped with these *coordinate charts*, G becomes a *Lie group*, in which the group multiplication operation $G \times G \to G$ becomes a differentiable map.
- To discuss what *tangent vectors*, *tangent spaces*, and *derivatives of functions* $f : G \to \mathbb{R}$" might mean on a group G equipped with differentiable structure, which might be realized as curvilinear hypersurface of high dimension. This is the realm of differential geometry.
- The result is a construct G that has both geometric and algebraic aspects. Lie groups can be quite complicated, but deal with them we must because they turn up at the heart of modern physics (and other fields), for instance as the symmetry groups that govern: the interactions of subatomic particles; the behavior of the cosmos according to special and general relativity; and the seemingly paradoxical rules of quantum mechanics. Even the periodic table of chemistry can in the end be deduced as a consequence of the mathematics associated with that classic example of a Lie group — the *special orthogonal group* SO(3) of rotations in 3-dimensional space.
- Finally, we will show that the tangent space to a Lie group G at its identity element, which at first sight is just a vector space of the same dimension as the coordinate charts that cover G, acquires algebraic structure induced by the multiplication law in G and becomes what is known as a *Lie algebra*.

  The important point is that this Lie algebra is a *linear* structure, much easier to deal with than G, and yet it encodes almost all the information to completely reconstruct and understand the Lie group from which it was derived.

Organization of the Text

The handling of exercises is somewhat unconventional. Many are placed within the main text, as the topics they address first occur. Each chapter ends with an extensive set of section-by-section Additional Exercises. Some recap the main topics of of each section; others are longer and intended to be more challenging; each begins with a block of true/false questions, which students often find more challenging than you might expect.

There are a few unconventional notations, explained as they are introduced:

- We often write $|V|$ for the dimension $\dim(V)$, and $R(T), K(T)$ for the range and kernel of T, respectively, when this is convenient.

- A special symbol ɫ is used to distinguish the constant function (or polynomial) everywhere equal to 1, from the scalar 1.

The symbol $\mathbb{K}$ indicates a generic ground field.

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Linear Algebra II

FREDERICK P. GREENLEAF AND SOPHIE MARQUES

This book is the second of two volumes on linear algebra for graduate students in mathematics, the sciences, and economics, who have: a prior undergraduate course in the subject; a basic understanding of matrix algebra; and some proficiency with mathematical proofs. Both volumes have been used for several years in a one-year course sequence, Linear Algebra I and II, offered at New York University's Courant Institute.

The first three chapters of this second volume round out the coverage of traditional linear algebra topics: generalized eigenspaces, further applications of Jordan form, as well as bilinear, quadratic, and multilinear forms. The final two chapters are different, being more or less self-contained accounts of special topics that explore more advanced aspects of modern algebra: tensor fields, manifolds, and vector calculus in Chapter 4 and matrix Lie groups in Chapter 5. The reader can choose to pursue either chapter. Both deal with vast topics in contemporary mathematics. They include historical commentary on how modern views evolved, as well as examples from geometry and the physical sciences in which these topics are important.

The book provides a nice and varied selection of exercises; examples are well-crafted and provide a clear understanding of the methods involved.

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