

C O U R A N T

31

S . R . S . V A R A D H A N

LECTURE
NOTES

Harmonic Analysis

American Mathematical Society
Courant Institute of Mathematical Sciences



Harmonic Analysis

Courant Lecture Notes in Mathematics

Executive Editor

Jalal Shatah

Managing Editor

Paul D. Monsour

Production Editor

Neelang Parghi

Copy Editor

Holli Chopra

S.R.S. Varadhan

Courant Institute

31 Harmonic Analysis

Courant Institute of Mathematical Sciences
New York University
New York, New York

American Mathematical Society
Providence, Rhode Island

For additional information and updates on this book, visit
www.ams.org/bookpages/cln-31

Library of Congress Cataloging-in-Publication Data

Names: Varadhan, S. R. S., author.

Title: Harmonic analysis / S.R.S. Varadhan, Courant Institute of Mathematical Sciences, New York University.

Description: Providence, Rhode Island : American Mathematical Society, [2022] | Series: Courant lecture notes, 1529-9031 ; volume 31 | Includes bibliographical references and index.

Identifiers: LCCN 2021044384 | ISBN 9781470465070 (paperback) | ISBN 9781470468934 (ebook)

Subjects: LCSH: Harmonic analysis. | AMS: Harmonic analysis on Euclidean spaces – Instructional exposition (textbooks, tutorial papers, etc.) pertaining to harmonic analysis on Euclidean spaces. | Harmonic analysis on Euclidean spaces – Harmonic analysis in one variable. | Harmonic analysis on Euclidean spaces – Harmonic analysis in several variables. | Probability theory and stochastic processes – Stochastic processes – Prediction theory (aspects of stochastic processes).

Classification: LCC QA403 .V35 2022 | DDC 515/.2433–dc23/eng/20211102

LC record available at <https://lcn.loc.gov/2021044384>

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy select pages for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for permission to reuse portions of AMS publication content are handled by the Copyright Clearance Center. For more information, please visit www.ams.org/publications/pubpermissions.

Send requests for translation rights and licensed reprints to reprint-permission@ams.org.

© 2022 by the author. All rights reserved.

Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines established to ensure permanence and durability.

Visit the AMS home page at <https://www.ams.org/>

10 9 8 7 6 5 4 3 2 1 27 26 25 24 23 22

Contents

Preface	vii
Chapter 1. Fourier Series	1
1.1. Introduction	1
1.2. Convergence of Fourier series	3
1.3. Special case $\mathbf{p} = \mathbf{2}$	7
1.4. Higher dimensions	7
1.5. Maximal inequality	8
1.6. Exercises	11
Chapter 2. Fourier Transforms on \mathbf{R}^d	13
2.1. Smooth rapidly decaying functions	13
2.2. Exercises	18
Chapter 3. Singular Integrals	21
3.1. Interpolation theorems	21
3.2. Weak type inequality	24
3.3. Exercises	29
Chapter 4. Riesz Transforms on \mathbf{R}^d	31
4.1. Singular integrals on \mathbf{R}^d	31
4.2. Riesz kernels	36
4.3. Exercises	39
Chapter 5. Sobolev Spaces	41
5.1. Generalized derivatives	41
5.2. Approximation theorems	42
5.3. Embedding theorems	43
5.4. Trace and extension theorems	47
5.5. Fractional derivatives	49
5.6. Generalized functions	50
5.7. Exercises	51
Chapter 6. Hardy Spaces	53
6.1. Stationary Gaussian processes	53
6.2. Hardy spaces	53
6.3. Inner and outer functions	55
6.4. Connection to prediction theory	61
6.5. Exercises	65

Chapter 7. Bounded Mean Oscillation	67
7.1. Functions of bounded mean oscillation	67
7.2. Duality of \mathbf{BMO} and \mathbf{H}_1	69
7.3. Exercises	74
Chapter 8. Elliptic PDEs	75
Chapter 9. Banach Algebras and Wiener's Theorem	83
Chapter 10. Compact Groups	85
10.1. Haar measure	85
10.2. Representations of a group	86
10.3. Representations of a compact group	88
Chapter 11. Representations of Two Compact Groups	93
11.1. Representations of the permutation group	93
11.2. Representations of $\mathbf{SO}(3)$	96
References	99
Index	101

Preface

Harmonic analysis is the study of functions by decomposing them into components that are special functions. A prime example is decomposing a periodic function into a linear combination of sines and cosines. The subject is vast, and these notes cover only the material that was dealt with in the course given at the Institute in 2000 and 2019.

References

- (1) Adams, R. A.; Fournier, J. J. F. *Sobolev spaces*. Second edition. Pure and Applied Mathematics (Amsterdam), 140. Elsevier/Academic Press, Amsterdam, 2003.
- (2) Bergh, J.; Löfström, J. *Interpolation spaces. An introduction*. Grundlehren der Mathematischen Wissenschaften, 223. Springer, Berlin–New York, 1976.
- (3) Dym, H.; McKean, H. P. *Séries et intégrales de Fourier*. Nouvelle Bibliothèque Mathématique, 13. Cassini, Paris, 2016.
- (4) Evans, L. C. *Partial differential equations*. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.
- (5) Folland, G. B. *Fourier analysis and its applications*. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1992.
- (6) Folland, G. B. *A course in abstract harmonic analysis*. Second edition. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2016.
- (7) Simon, B. *Representations of finite and compact groups*. Graduate Studies in Mathematics, 10. American Mathematical Society, Providence, RI, 1996.
- (8) Stein, E. M. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, 30. Princeton University Press, Princeton, NJ, 1970.
- (9) Stein, E. M. *Topics in harmonic analysis related to the Littlewood-Paley theory*. Annals of Mathematics Studies, 63. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1970.
- (10) Stein, E. M.; Weiss, G. *Introduction to Fourier analysis on Euclidean spaces*. Princeton Mathematical Series, 32. Princeton University, Princeton, NJ, 1971.
- (11) Stein, E. M. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*. Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.
- (12) Stein, E. M.; Shakarchi, R. *Fourier analysis. An introduction*. Princeton Lectures in Analysis, 1. Princeton University Press, Princeton, NJ, 2003.

Index

- a priori estimate, 78
- Banach algebras, 83
- BMO, 67
- bounded mean oscillation, 67
- character of a representation, 89
- compact groups, 85
- convergence of Fourier series, 3
- covering lemma, 32
- Duality of **BMO** and \mathbf{H}_1 , 69
- elliptic PDEs, 75
- embedding theorems, 43
- extension theorems, 47
- Fejér sum, 4
- Fourier coefficients, 2
- Fourier series, 1, 7
- Fourier transforms on \mathbf{R}^d , 13
- fractional derivatives, 49
- Gagliardo-Nirenberg-Sobolev inequality, 45
- Generalized derivatives, 41
- generalized functions, 50
- Haar measure, 85
- Hardy spaces, 53
- Hardy-Littlewood maximal function, 8, 10
- heat equation, 1
- infinitesimal method, 96
- inner and outer functions, 55
- interpolation theorems, 21
- John-Nirenberg theorem, 67
- Marcinkiewicz, 21
- maximal inequality, 8
- maximum principle, 80
- Peter-Weyl, 88
- Plancherel-Parseval identities, 7
- Poisson kernel, 10
- Poisson maximal function, 10
- Poisson representation formula, 54
- Poisson sum, 9
- positive definite functions, 17
- prediction theory, 61
- representations of a compact group, 88
- representations of a group, 86
- representations of $\mathrm{SO}(3)$, 96
- representations of the permutation group, 93
- Riemann-Lebesgue lemma, 2
- Riesz kernels, 36
- Riesz transforms, 30
- Riesz-Thorin interpolation, 23
- Schur's lemma, 88
- singular integrals on \mathbf{R}^d , 30
- Sobolev spaces, 41
- stationary Gaussian processes, 53
- topological group, 85
- trace theorems, 47
- Vitali covering lemma, 8
- weak type inequality, 24
- Wiener's theorem, 83
- Young tableau, 93
- Young's inequality, 43
- Young's theorem, 16

Published Titles in This Series

- 31 **S.R.S. Varadhan**, Harmonic Analysis, 2022
- 30 **Frederick P. Greenleaf and Sophie Marques**, Linear Algebra II, 2020
- 29 **Frederick P. Greenleaf and Sophie Marques**, Linear Algebra I, 2019
- 28 **László Erdős and Horng-Tzer Yau**, A Dynamical Approach to Random Matrix Theory, 2017
- 27 **S. R. S. Varadhan**, Large Deviations, 2016
- 26 **Jerome K. Percus and Stephen Childress**, Mathematical Models in Developmental Biology, 2015
- 25 **Kurt O. Friedrichs**, Mathematical Methods of Electromagnetic Theory, 2014
- 24 **Christof Schütte and Marco Sarich**, Metastability and Markov State Models in Molecular Dynamics, 2013
- 23 **Jerome K. Percus**, Mathematical Methods in Immunology, 2011
- 22 **Frank C. Hoppensteadt**, Mathematical Methods for Analysis of a Complex Disease, 2011
- 21 **Frank C. Hoppensteadt**, Quasi-Static State Analysis of Differential, Difference, Integral, and Gradient Systems, 2010
- 20 **Pierpaolo Esposito, Nassif Ghossoub, and Yujin Guo**, Mathematical Analysis of Partial Differential Equations Modeling Electrostatic MEMS, 2010
- 19 **Stephen Childress**, An Introduction to Theoretical Fluid Mechanics, 2009
- 18 **Percy Deift and Dimitri Gioev**, Random Matrix Theory: Invariant Ensembles and Universality, 2009
- 17 **Ping Zhang**, Wigner Measure and Semiclassical Limits of Nonlinear Schrödinger Equations, 2008
- 16 **S. R. S. Varadhan**, Stochastic Processes, 2007
- 15 **Emil Artin**, Algebra with Galois Theory, 2007
- 14 **Peter D. Lax**, Hyperbolic Partial Differential Equations, 2006
- 13 **Oliver Bühler**, A Brief Introduction to Classical, Statistical, and Quantum Mechanics, 2006
- 12 **Jürgen Moser and Eduard J. Zehnder**, Notes on Dynamical Systems, 2005
- 11 **V. S. Varadarajan**, Supersymmetry for Mathematicians: An Introduction, 2004
- 10 **Thierry Cazenave**, Semilinear Schrödinger Equations, 2003
- 9 **Andrew Majda**, Introduction to PDEs and Waves for the Atmosphere and Ocean, 2003
- 8 **Fedor Bogomolov and Tihomir Petrov**, Algebraic Curves and One-Dimensional Fields, 2002
- 7 **S. R. S. Varadhan**, Probability Theory, 2001
- 6 **Louis Nirenberg**, Topics in Nonlinear Functional Analysis, 2001
- 5 **Emmanuel Hebey**, Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities, 2000
- 3 **Percy Deift**, Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach, 2000
- 2 **Jalal Shatah and Michael Struwe**, Geometric Wave Equations, 2000
- 1 **Qing Han and Fanghua Lin**, Elliptic Partial Differential Equations, Second Edition, 2011

Harmonic Analysis

S. R. S. VARADHAN

Harmonic Analysis is an important tool that plays a vital role in many areas of mathematics as well as applications. It studies functions by decomposing them into components that are special functions. A prime example is decomposing a periodic function into a linear combination of sines and cosines. The subject is vast, and this book covers only the selection of topics that was dealt with in the course given at the Courant Institute in 2000 and 2019. These include standard topics like Fourier series and Fourier transforms of functions, as well as issues of convergence of Abel, Feier, and Poisson sums. At a slightly more advanced level the book studies convolutions with singular integrals, fractional derivatives, Sobolev spaces, embedding theorems, Hardy spaces, and BMO. Applications to elliptic partial differential equations and prediction theory are explored. Some space is devoted to harmonic analysis on compact non-Abelian groups and their representations, including some details about two groups: the permutation group and $SO(3)$.

The text contains exercises at the end of most chapters and is suitable for advanced undergraduate students as well as first- or second-year graduate students specializing in the areas of analysis, PDE, probability or applied mathematics.



For additional information
and updates on this book, visit

www.ams.org/bookpages/cln-31



NEW YORK UNIVERSITY

ISBN 978-1-4704-6507-0



9 781470 465070

CLN/31