The Logarithmic Potential: Discontinuous Dirichlet and Neumann Problems

Griffith Conrad Evans
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to

VITO VOLterra
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PREFACE

This small treatise is an outgrowth of a study of Stieltjes integrals and potential theory which the author published in the 1920 volume of the Rice Institute Pamphlet, and a needed revision and development of the last part of that essay in the direction indicated by three notes which appeared in 1923, in the Comptes rendus des séances de l'Academie des Sciences. Two of these were written in conjunction with my colleague, Professor H. E. Bray. The work gives a unified treatment of the basis of the theory of Laplace's equation in two dimensions, suitable, it is hoped, for graduate students of a moderate degree of advancement, and is intended to be of service in the development of the theory of partial differential equations of elliptic type. These developments are generating a compound of two of the most important elements of modern analysis—the concepts of Lebesgue on the one hand, and of Volterra on the other.

An earlier form of part of the treatise was given in lectures at the Rice Institute in the academic year 1924-25, in connection with a course in the theory of functions of a real variable, and at the University of Chicago during the Summer Quarter of 1925. Chapter VII furnished the substance of an invited discourse at the meeting of the Southwestern Section of the American Mathematical Society in November, 1926.

The author is much indebted to Professor O. D. Kellogg, who has seen a large portion of the manuscript, and aided with kindly criticism, to Professor Bray, who has read the proof sheets, and, finally, to the American Mathematical Society, through whose generosity the publication is possible.

Houston, Texas.
June, 1927.

Griffith C. Evans.
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This book studies fundamental properties of the logarithmic potential and their connections to the theory of Fourier series, to potential theory, and to function theory. The material centers around a study of Poisson's integral in two dimensions and of the corresponding Stieltjes integral. The results are then extended to the integrals in terms of Green's functions for general regions. There are some thirty exercises scattered throughout the text. These are designed in part to familiarize the reader with the concepts introduced, and in part to complement the theory. The reader should know something of potential theory, functions of a complex variable, and Lebesgue integrals. The book is based on lectures given by the author in 1924–1925 at the Rice Institute and at the University of Chicago.