Linear Transformations in Hilbert Space and Their Applications to Analysis

Marshall Harvey Stone
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FOREWORD

When I began in the summer of 1928 to study the theory of linear transformations in Hilbert space, it was my intention to present the results in some of the current journals. Various circumstances have led, however, to the preparation of this volume, in which I have attempted to include not only my own independent contributions but also a substantial portion of the existing material bearing on the subject. The lack of English-language works dealing with the theory of Hilbert space appeared to be an adequate reason for planning a detailed treatment which would start with the foundations and carry the development as far as possible in every direction. It has accordingly been my object to provide a treatise of this character, with serious claims to completeness—one which will be, if my hopes are realized, a useful handbook both for the student and for the investigator. In carrying out this plan, I have confined myself strictly to the theory of Hilbert space, arbitrarily excluding any reference to the various related and similar theories of other types of space. The adoption of this course was merely an unfortunate consequence of the necessity of keeping the length of this book within reasonable limits, and does not imply any personal view or judgment concerning the interest and importance of other theories. I should be the first to urge the reader to consult discussions of such cognate topics.* Considerations of space have also

* For mathematical and bibliographical information, see Volterra, The Theory of Functionals and Integro-Differential Equations, London, 1930; —Hildebrandt, Bulletin of the American Mathematical Society, 37 (1931), pp. 185–212. There are several books in preparation which will be of interest in this connection: Mrs. Pell-Wheeler will contribute a volume to this Collo-
led to the omission of two chapters for which provision had originally been made. One of these, dealing with groups of transformations in Hilbert space, was completed in May, 1930, and will appear separately in the course of time. The other, outlining the remarkable applications of the abstract theory developed in the present volume to modern atomic physics, was never written. Although this chapter would have been purely expository in nature, I regret that it could not be included, since the point of view which it would have set forth in some detail has already proved so fruitful in the study of the atom and promises still more profound results in the future.*

The existence of the splendid article by Hellinger and Toeplitz in the Encyklopädie der Mathematischen Wissenschaften, 2nd, has spared me the exacting labor of preparing a bibliography of the voluminous literature prior to 1924. I have taken some pains, however, to give references to the major contributions since that date.† While I have concerned myself very little

quium Series (compare the synopsis of her Colloquium Lectures, Bulletin of the American Mathematical Society, 33 (1927), pp. 664-665); in the Introduction to his book, Les espaces abstraits, Paris, 1928, Fréchet has promised a second volume dealing with functional analysis; and, I am informed, the researches of E. H. Moore and his school, which have long been practically unavailable to the mathematical public, will soon be presented in book form. A French translation of Banach’s Teorja Operacij, Tom I: Operacje liniowe, Warszawa, 1931, is about to appear.


† The recent paper of J. v. Neumann, Annals of Mathematics, (2) 33 (1932), pp. 294-310, came to my attention too late to be cited. This paper throws a great deal of light on Theorems 2.9, 2.10, 2.26, 8.18, 9.5, 10.10 and Definition 8.3. For instance, the hypothesis of Theorem 2.26 can be weakened to read “If T is a transformation whose domain is 8 and if its adjoint T* has domain everywhere dense in 8”; an independent proof of the modified theorem was communicated to me by Professor J. D. Tamarkin.
with questions of history or of priority, I wish to acknowledge in the most cordial spirit my scientific debt to J. v. Neumann. The initial impetus of my interest came from reading some of v. Neumann's early and still incomplete work, which was described in the Göttinger Nachrichten, 1927, pp. 1-55, footnotes 12 and 27, to which I had access, but which was never published. Thereafter, I worked independently, the results announced in the Proceedings of the National Academy of Sciences, 15 (1929), pp. 198-200, pp. 423-425, and 16 (1930), pp. 172-175, being obtained without further knowledge of his progress along the same or similar lines. I have been only too glad to improve the final presentation of my own investigations by the continual use of v. Neumann's various memoirs on the theory of transformations in Hilbert space. While it is scarcely necessary to point out that this recent work is a natural continuation of that begun by Hilbert and his school, I wish to emphasize the important rôle played by the contributions of F. Riesz in preparing the ground for a successful consideration of non-bounded transformations. The concepts which Riesz developed in his book, Les systèmes d'équations linéaires à une infinité d'inconnues, Paris, 1913, marked the introduction of a new point of view and of new methods, without which progress might well have been retarded; their influence can be traced throughout the development of the theory given in these pages.

With a view to making the book useful as a work of reference, I have adopted the practice of stating all important definitions and theorems in italics and numbering them serially by chapters: thus Theorem 5.12 is the twelfth theorem in Chapter V, Definition 8.2 the second definition in Chapter VIII. At the same time, I have avoided, so far as possible, any elaborate system of cross-references in either the text or the foot-notes, so that the reader need not correlate a mass of widely scattered material when he is interested in a particular topic. The table of contents and the index are designed to serve as adequate guides to the various subjects treated. In order to compress the material into the compass of six hundred
odd pages, it has been necessary to employ as concise a style as is consistent with completeness and clarity of statement, and to omit numerous comments, however illuminating, which will doubtless suggest themselves to the reader as simple corollaries or special cases of the general theory.

It is a great pleasure to express my gratitude to the many friends who have encouraged or aided me in the task of preparing this book. I wish, above all, to thank Professor J. D. Tamarkin of Brown University, whose interest fostered the project from its inception and whose patient criticism, freely and unselfishly given, has guided it to maturity. More formal thanks are owed to the administrators of the Milton Fund of Harvard University, who, during my connection with the University, granted moneys for the preparation of the manuscript; and to the Committee on Colloquium Publications of the American Mathematical Society, who have generously honored my work by applying to it the policy of publishing material which has not been presented in the form of Colloquium Lectures before the Society.

New Haven, Connecticut, April, 1932.  
M. H. Stone.
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