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# Algebraic Functions

Gilbert Ames Bliss

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## Preface

The theory of algebraic functions has been developed by three different methods which have been designated as transcendental, algebraic-geometric, and arithmetic. Two very illuminating comparisons of these theories have been made by Hensel and Landsberg [28, pages 694–702]* and Emmy Noether [37]. The transcendental method had its origin in a paper by Abel [1] in 1826, in which he announced the remarkable generalization of the addition formulas for elliptic integrals which is now called Abel's theorem. The theory has been greatly enriched by many writers, but especially by Riemann [11]. It is called transcendental because in it Abelian integrals play the fundamental role. The algebraic-geometric theory is a theory of algebraic plane curves. Early expositions were given by Clebsch and Gordan [3] in 1863–66, and by Brill and Noether [17] in 1871. A more recent account is that of Severi [39, 45] in 1921 and 1926, in which much emphasis is placed upon the properties of linear families of curves and their intersections. The title "arithmetic" is applied to a group of theories which differ greatly in detail but which have in common as central features the construction and analysis of the rational functions which are the integrands of Abelian integrals. One of the earliest suggestions of such a theory is found in a paper [7] which Kronecker presented to the Berlin Academy in 1862 but published first in 1881. More elaborate theories in the arithmetic group are those of Weierstrass [27] in his lectures of 1875–6, of Dedekind and Weber [9] in 1882, and of Fields [30] in 1906. The method of Weierstrass is an application to algebraic functions of his theory of analytic functions, and somewhat the same remark would apply to the method of Fields. Dedekind and Weber, however, emphasized the analogies between the theories of algebraic functions and algebraic numbers. Their methods have been elaborated and improved by Hensel and Landsberg in their book on algebraic functions [28] published in 1902, and in later memoirs.

In the following pages, after an introductory Chapter I, I have endeavored to give in Chapters II and III a concise but readable introduction to the arithmetic theory of algebraic functions. My purpose was

* The numbers in square brackets here and elsewhere in the text refer to the list of references at the end of this book.

to attain as directly as possible the proofs of the existence, and the methods of construction, of the integrands of the three kinds of elementary integrals, and the theorem of Riemann-Roch. These are fundamental results which it has always seemed to me desirable to have available early, rather than buried deeply, in the text of the theory. The methods used are those of Hensel and Landsberg with many variations. I have, for example, discarded almost entirely the nomenclature and use of the ideals of Dedekind and Weber, which are of great interest, but which are auxiliary rather than essential in the development of the theory.

An introduction to the methods of the transcendental theory is given in Chapters IV, V, and VI, the first of which is devoted to Riemann surfaces and Cauchy's theorem, the second to the definition and properties of Abelian integrals, and the third to the famous theorem of Abel which inaugurated the transcendental theory. An advantage of this order of presentation of the arithmetic and transcendental theories is that no preliminary transformation simplifying the singularities of the fundamental algebraic curve is required.

Chapters VII and VIII are devoted to birational transformations. In the former, fundamental properties and some simple applications of such transformations to the reduction of special algebraic curves to normal forms are explained. Chapter VIII is devoted to two famous transformation theorems. The first of these states that every algebraic curve can be reduced by a Cremona transformation to one having no singular points other than multiple points with distinct tangents, and the second asserts that by a less special birational transformation every such curve can be transformed into another having only double points with distinct tangents. These theorems have been important for the transcendental and algebraic-geometric methods, because these methods have a much simpler aspect when the only singularities of the algebraic curve under consideration are multiple points with distinct tangents.

The literature of the second of the transformation theorems mentioned above is very large. Many of the proofs of the theorem are incomplete, and very few of them have escaped amplification or criticism. It has not been generally recognized in the literature that there are really two theorems involved, one for the function-theoretic and one for the projective plane. In a paper [43] published in 1923 I have given a history of the theorem and have emphasized these remarks. In Chapter VIII below a proof of the theorem for the function-theoretic plane is given which was suggested in the paper of 1881 by Kronecker, and completed in 1902 by Hensel and Landsberg in their book on algebraic functions mentioned above [pages 402–9]. The theorem seems to me distinctly more difficult to prove for the projective plane. In a paper [41] published in 1922 I showed how the reasoning of Kronecker, and Hensel and Landsberg, can be extended to apply to the projective case also. The proof is reproduced in improved form in Chapter VIII. I find great differences of opinion among mathematicians concerning the validity and the advantages of the many different methods of proving these transformation theorems. The method given here has at any rate an especial interest from the standpoint of the arithmetic theories of algebraic functions.

Chapter IX below is devoted to the inversion problem for algebraic curves of genus zero or unity, and to the relations between the theories of elliptic functions and the rational functions associated with an algebraic curve of genus one. I have regretted the necessity of omitting the theory of the inversion problem for greater values of the genus. The presentation of it in a satisfactory manner would require a much larger book than this.

Illustrative examples have great value for a reader who is orienting himself in a mathematical theory for the first time. Chapter X is devoted to such examples, which may be studied in connection with the text from Chapter II on. Not all of these examples are merely exercises. The elliptic and hyperelliptic cases described in Section 70 have of course great importance and many applications. In the final sections of the chapter the methods of Baur [14, 19] for algebraic equations $f(x, y)=0$ of the third degree in $y$ are explained in detail. These have the advantage of requiring for their applications no more complex algebraic mechanisms than the highest common divisor process. Their generality is illustrated by the fact that they are applicable also to equations of the fourth degree in $x$ and $y$ after a suitable transformation.

Following Chapter X is a list of books and memoirs to which references are made above and elsewhere by numbers in square brackets. At the end of each chapter a brief note indicates reading which may be helpful in connection with the material presented in that particular chapter.

The book as a whole is introductory in character and not a comprehensive treatise. It is an account of lectures on algebraic functions which I have given at the University of Chicago a number of times, the most recent one being the Summer Quarter of 1931. In that year I prepared for students a mimeographed set of notes from which the following pages have been developed. In their preparation I have been ably assisted by Dr. M. R. Hestenes. He has read the manuscript with care and has made many valuable suggestions. I record here my appreciation of his helpful interest.

In conclusion I wish to acknowledge with gratitude the interest of the Editors of the Colloquium Publications of the American Mathematical Society, and the assistance of the National Research Council, which have made possible the publication of this book.

G. A. Bliss

The University of Chicago, 1932

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The list below is a list of books which are of interest on account of the influence which they have had in the development of the theory of algebraic functions, or on account of their usefulness as presentations of parts of the theory in the preceding pages, or both. A few titles of separate papers have been added, most of which have some close connection with the preceding text. A complete bibliography would be a voluminous affair. For further references the reader is referred especially to the articles numbered 17, 22, 23, 26, 29, 37, 40, and 46 below.


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Weierstrass, iii, 176–7

*This book, immediately striking for its conciseness, is one of the most remarkable works ever produced on the subject of algebraic functions and their integrals. The distinguishing feature of the book is its third chapter, on rational functions, which gives an extremely brief and clear account of the theory of divisors....*

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