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Fourier Transforms in the Complex Domain

R. C. Paley
N. Wiener

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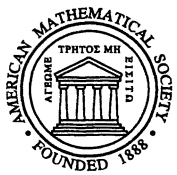
Raymond E. A. C. Paley
Norbert Wiener



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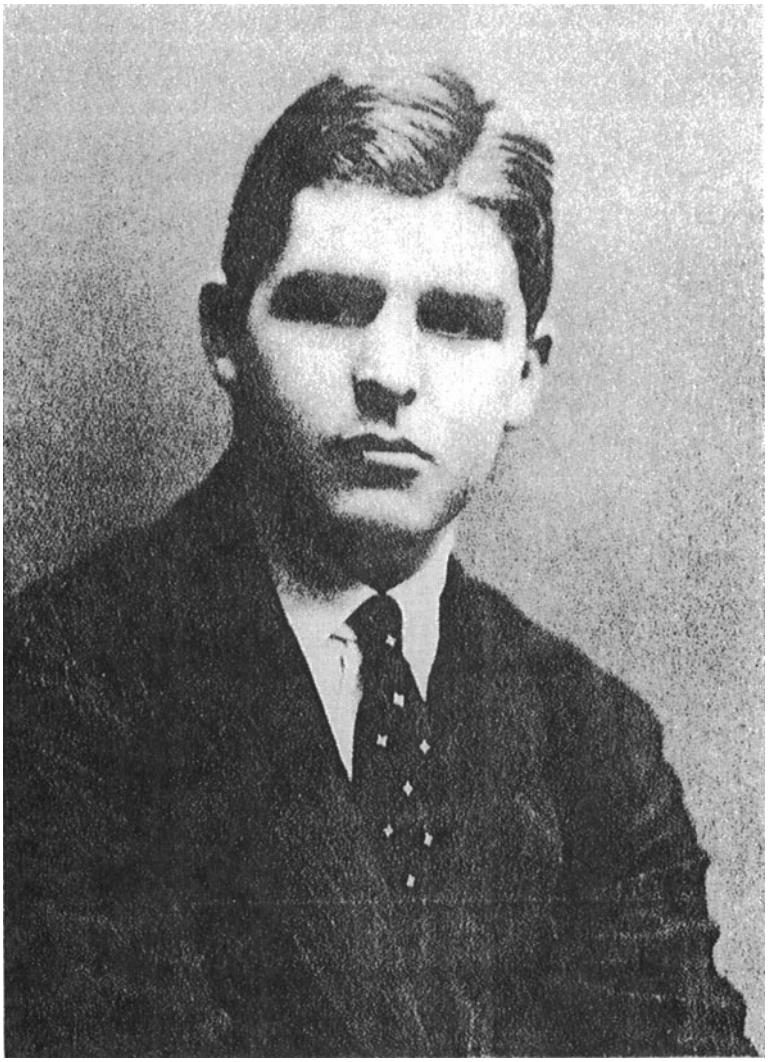

R. E. A. C. PALEY, 1907–1933

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Dedicated by the surviving author

to

PROFESSORS G. H. HARDY *and* J. E. LITTLEWOOD

the teachers of us both

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Preface

The present book represents a definitive statement of the results obtained by the late Mr. R. E. A. C. Paley and myself during Mr. Paley's year as Rockefeller Fellow at the Massachusetts Institute of Technology (1932–1933). Mr. Paley was killed on April 7 in a skiing accident in the Canadian Rockies, during a short vacation which he had taken from our joint work. I have written elsewhere of the great loss to mathematics by his death; here let me only state the condition in which our joint work was left. Our method of collaboration had been most informal. We had worked together with a blackboard before us, and when we had covered it with our joint comments, one or the other would copy down what was relevant, and reduce it to a preliminary written form. Most of our work went through several versions, in writing which both authors took part. Even in that part of the research committed to writing since Mr. Paley's death, it is completely impossible to determine how much is new and how much is a reminiscence of our many conversations.

A part of our work was published in the form of a series of notes in the Transactions of the American Mathematical Society. This work covered a great variety of topics, but was unified by the central idea of the application of the Fourier transform in the complex domain. I had long been convinced of the importance of the Fourier-Mellin transforms as a tool in analysis. Their introduction is of course no novelty, but I know of no systematic development of their methodical use. Perhaps the nearest approach to such a development is to be found in the researches of H. Bohr, Jessen and Besicovitch on almost periodic functions in the complex domain. However, nobody seems to have realized anything like the scope of the method. With its aid, we were able to attack such diverse analytic questions as those of quasi-analytic functions, of Mercer's theorem on summability, of Milne's integral equation of radiative equilibrium, of the theorems of Münz and Szász concerning the closure of sets of powers of an argument, of Titchmarsh's theory of entire functions of semi-exponential type with real negative zeros, of trigonometric interpolation and developments in polynomials of the form

$$\sum_{1}^{N} A_n e^{i\lambda_n x},$$

of lacunary series, of generalized harmonic analysis in the complex domain, of the zeros of random functions, and many others. We came to believe that an analytic method of such scope is entitled to an independent treatise.

The American Mathematical Society has done me the honor of requesting me to deliver the Williamstown Colloquium Lectures for 1934. While such lectures have not previously been an account of collaborative work, my best available work has been collaborative, and I have offered it for the lectures in question.

I wish to thank the American Mathematical Society for its invitation, and for its acceptance of our plans. I wish to thank my students, Messrs S. S. Saslaw, H. Malin, and N. Levinson, for most valuable and painstaking work of revision, compilation, and criticism. Mr. Levinson, in particular, has added much to the content of Chapter I. I wish to thank my colleague, Professor Eberhard Hopf, for permission to incorporate into this book the material of §17, which was our joint work. Furthermore, in my own name and in the name of my dead co-author, I wish to thank Professor J. D. Tamarkin of Brown University for his untiring encouragement, advice, and criticism, without which this book would not have come into existence.

NORBERT WIENER.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
CAMBRIDGE, MASSACHUSETTS, MARCH 1, 1934.

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