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Joseph Miller Thomas

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To
D. M.

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PREFACE

The primary purpose of this book is to develop the theory of systems of partial differential equations and that of pfaffian systems so as to exhibit clearly the relation between the two theories. The questions treated concern almost exclusively the existence of solutions and methods of approximating them rather than their properties, whose study seems to belong to the theory of functions.

In writing the book the author has been guided by a desire for generality in results and conciseness in subject matter and proofs. As a consequence, the postulational method seemed to force itself upon him. Roughly, the plan has been to take a few existence theorems as postulates and construct the theory upon them. A consistency proof is included by proving the postulates in particular cases. The original plan included extensions of the consistency proofs, but the pressure of other duties prevented carrying this out.

The ideas and nomenclature of modern algebra, as developed, for instance, in van der Waerden's admirable treatise, have been freely used. Some modifications of certain topics, essential for our purposes, have been included, but no systematic development of the theory of commutative polynomial rings has been made. On the other hand, the theory of a certain non-commutative polynomial ring, called here a Grassmann ring, is developed in detail from the postulates in Chapter III, which together with Chapter IV develops ideas introduced by Grassmann and brought to such a high degree of perfection by Cartan. A combination of Cartan's notation, the tensor calculus, and modern algebraic concepts seems very effective. Incidentally, the results about determinants and linear dependence, which are needed, can be proved directly from the postulates as readily as the manner of stating them in the literature can be modified to fit the case in hand.

The treatment of the algebraic case is the author's. Although it has close connection through the highest common factor with Ritt's excellent discussion, which is based on the division algorithm, it differs radically in several respects from that work because of a difference in purpose and viewpoint. In the first place, the basis of our method is algebra, rather than analysis. Secondly, reducibility, which plays such a prominent rôle in Ritt's developments, is of little importance in ours. With existence theorems as our chief objective, the important thing for us is to eliminate multiple roots. A polynomial's having two factors, for example, does not prevent the application of the implicit function theorem, if the factors are distinct, and making that theorem applicable is the chief purpose of the reduction process. Incidentally, it might be well to point out that the term "reducible" has slightly different meanings in the two theories. The system y^2, which Ritt classes as irreducible, is reducible in ours.

Another feature of our treatment, which assumes its most elegant and satisfactory form in the algebraic case, although employed in the whole work, is the admission of the inequation on an equal footing with the equation. This, together with the use of resultants of all orders (subresultants), obviates the necessity of making the preliminary linear transformation of the indeterminates, which is an essential step in Kronecker's method of solution of algebraic systems.

Finally, the algebraic case furnishes the model for treating the elimination problem for systems of functions. This is done in Chapter VIII. The method is subject to certain limitations. First, there is no algorithm for determining the zeros of an analytic function in a given region. The difficulty of removing this restriction can be appreciated if the zeros of the Riemann ζ-function are cited. Second, there may exist zeros which are not the centers of regions where assumption W is true. These zeros may be termed singular. Their determination and study seem destined to remain for some time a highly complex problem, only to be solved in special cases by special methods. In this respect they resemble the solutions of a system of partial differential equations in the neighborhood of a singular point. In spite of these limitations, the general method of elimination given here seems to furnish a definite result, which is perhaps as satisfactory as can be obtained at present.

In addition to bringing Cartan's existence theorem for pfaffian systems into the scheme, Chapter IX shows clearly that it has limitations because it does not give the singular integral varieties unless substantially modified. The same chapter also gives what is believed to be the only method yet developed for finding and making a partial classification of the singular integral varieties. The method ultimately—and it seems essentially—depends on Riquier's fundamental researches.

In order not to interrupt the continuity of the development, the illustrative examples have been segregated in Chapter XI. The reader may find it convenient to study them at the appropriate place in the text.

The author has drawn freely from the work of Cartan, Goursat, and Janet, but he is particularly indebted to Riquier's treatise. The book also incorporates many suggestions made by students in his courses during the past nine years; the present neat statement of the rule of signs in Theorem 9.1, for example, was suggested by Mr. Alexander Makarov. The author is even more indebted to all those who have listened to his lectures for sustaining his interest in the subject by their sympathetic attention.

J. M. THOMAS

July, 1936

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The main goal of this book is to present the theory of systems of partial differential equations and the theory of Pfaffian systems so as to exhibit clearly the relations between them.

In presenting the theory of Pfaffian systems, the author develops, in detail, the theories of Grassmann algebras and rings with differentiation. In particular, following Grassmann and É. Cartan, he introduces and freely uses what is now known as a ring of differential forms with functional coefficients. In presenting the theory of systems of partial differential equations, the author concentrates on the existence of solutions and methods of approximating them, rather than on their properties. The relations and similarities of two theories are displayed through the systematic use of various versions of the elimination method.

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