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Charles N. Moore

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PREFACE

All methods for summing a divergent series which have come into general use may be classified as mean-value methods or convergence factor methods. Corresponding to any method of either type there can be constructed a formally equivalent method of the other type. The range of validity of the corresponding methods is in general approximately the same. Where there is a difference, it will be in favor of the convergence factor form. A well known instance of this is found in the relative ranges of applicability of Borel's integral definition and Borel's mean-value definition for the sum of a divergent series.

The relationship between a definition based on mean-values and a formally distinct definition in terms of convergence factors was one of the earliest problems studied in the field of divergent series, as is pointed out in the introduction to this book. Subsequently many other results concerning this relationship have been obtained by various workers in the field. Such results are now appropriately termed convergence factor theorems.

One may also be interested primarily in determining the conditions on a set of factors in order that they may preserve convergence for a convergent series or produce convergence for a summable series, when introduced into the terms of such series. The convergence factors used in defining the sum of a divergent series have this property and an additional one as well. To distinguish between the two cases we shall designate as convergence factors of type I those where only the property of maintaining or producing convergence is in question. Factors that may be used to obtain the sum of a series will be known as convergence factors of type II.

The aim of the present work is to give a systematic treatment of convergence factor theorems. Both types of convergence factors are considered, and the theory is developed for multiple series of any order as well as for simply infinite series. Through the use of Nörlund means in place of Cesàro means, the theory developed is considerably more general than that found in the existing literature. Many previous results thus appear as special cases of the theorems proved here, particularly the theorems in the third, fourth, and fifth chapters.

The writer takes pleasure in expressing here his appreciation of encouragement and assistance from various sources in connection with the preparation of this book. He was enabled to spend the entire academic year of 1934–1935 in residence at the Institute for Advanced Study through the aid of grants from the Institute and the Charles Phelps Taft Memorial Fund of the University of Cincinnati. During the period referred to, most of the basic research preliminary to the writing of the book was completed and a considerable portion of the manuscript was prepared. In this connection the excellent facilities for mathematical work available at Princeton and the stimulus of the scientific companionship to be found there were extremely helpful. The

preparation of the final draft of the manuscript was considerably facilitated by the careful and competent assistance of Dr. W. C. Mitchell, formerly Laws Fellow in Mathematics at the University of Cincinnati. Further help from the Taft Fund has been available to bear part of the expense for this work and part of the cost of publication. For all this assistance and for the coöperation of the American Mathematical Society in accepting this book for publication in the Colloquium Series the writer is deeply grateful.

CHARLES N. MOORE

THE UNIVERSITY OF CINCINNATI, 1937

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Fairly early in the development of the theory of summability of divergent series, the concept of convergence factors was recognized as of fundamental importance in the subject. One of the pioneers in this field was C. N. Moore, the author of the book under review.... Moore classifies convergence factors into two types. In type I he places the factors which have only the property that they preserve convergence for a convergent series or produce convergence for a summable series. In type II he places the factors which not only maintain or produce convergence but have the additional property that they may be used to obtain the sum or generalized sum of the series. This book gives a generalized systematic treatment of the theory of convergence factors of both types, for simply infinite series and for multiple series, convergent and summable....

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