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# Length and Area

Tibor Radó



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## PREFACE

The terms *curve*, *surface*, *length*, *area* have been used by many generations of mathematicians. And yet, no general agreement has been arrived at concerning the selection of precise formal definitions for these terms, as a basis of a general comprehensive theory. The theory of length and area which is presented in this book is based upon fundamental ideas of Lebesgue and Geöcze. As compared with other equally plausible and relevant approaches proposed during the past fifty years, this theory is distinguished by a high degree of completeness, achieved through the sustained efforts of many mathematicians. However, the number of difficult and fascinating problems is still large, and application of the results to other fields, especially to Calculus of Variations, is hardly begun. Hence the writer hopes that he was justified in emphasizing methods and results that seemed most relevant for further researches.

This book is an amplification of a set of four Colloquium lectures which the writer had the honor to deliver at the annual meeting of the American Mathematical Society in Chicago, November 1945. War-time conditions created delays and complications that were resolved in the most sympathetic manner by the officers of the Society. Several colleagues rendered invaluable help by reading and preparing for print the manuscript of this book. The Institute for Advanced Study in Princeton and the Ohio State University cooperated in providing leisure and material assistance. The writer wishes to express his appreciation of the cooperation he received from all these sources.

TIBOR RADÓ

COLUMBUS, OHIO,  
March 1946

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Radó's colloquium is a systematic treatment of Lebesgue theory, with an emphasis on the work of Morrey and of Radó and his students, especially in two dimensions. At the time, there were important current problems surrounding Lebesgue's theory for parameterized and unparameterized surfaces, which the book addresses. The exposition begins with reviews of Lebesgue integration and relevant topics in topology, including Fréchet equivalence, the approximation of monotone maps by homeomorphisms, Peano spaces, and a discussion of the topological index of maps into the plane. After a development of further ideas and tools from topology and measure theory, Radó addresses an essential question that equates two sorts of areas for surfaces represented by maps of a 2-cell or a 2-sphere into 3-space.

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