Topology of Manifolds

Raymond Louis Wilder
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PREFACE

The historical background of this work is sketched in Chapter I, section 6, and need not be repeated here. It should, however, be complemented by certain remarks of a more personal nature, particularly as regards the author's indebtedness to his mathematical colleagues.

It has become more or less apparent to students of cultural evolution that the genesis of a line of thought cannot be fixed either in chronological fashion or bibliographically. If proper evidence were on record, an idea which seems to emanate at a fixed date or in a particular work would be found upon analysis to be only the end product of a collection of prior ideas; the "originator" of the idea being only the medium through which these latter ideas achieve their synthesis. Even the particular individuality of the "originator" is probably not of paramount importance; of importance is the perennial presence of the "creative" mind, ready to receive the stimuli. Can anyone doubt that the calculus would have evolved even though Leibnitz and Newton had taken up farming instead of science? Simultaneous announcement of "discoveries" by contemporaries, often widely separated, is not a rare occurrence.

It is fitting, then, for an author to attempt to place his work in its proper setting amongst past and contemporary influences. This is the object of the historical remarks in Chapter I. But these formal remarks only partially fill in the picture. On the more personal side, I wish to express my indebtedness to Professor R. L. Moore, under whose tutelage I received a thorough grounding in point set theory. It was during my early contacts with him that I came to realize the vacuum in our knowledge of the set-theoretic structure of the $n$-cell, particularly the lack of a topological characterization. Later, through personal contacts with Professor Paul Alexandroff in 1928, I became convinced (a conviction which he obviously shared) that the problem of the $n$-cell demanded new tools, especially the extension to general spaces of the theory of connectivity (homology). Acknowledgements are also due to Professor Eduard Čech (whose theory of general homology is used herein), who visited the United States in 1934–35 and from whom I gained much stimulation and personal encouragement. I am also grateful to the Institute for Advanced Study for making possible a year's uninterrupted research in 1933–34, during which the present investigations on manifolds were initiated; and to the John Simon Guggenheim Memorial Foundation for the grant of a fellowship in 1940–41. It was during the latter period that the euclidean form of many of the results given in Chapters X–XII were found.

As regards the end result—the book itself—it cannot be emphasized too strongly that what is presented herewith is only a beginning. It is only those properties of manifolds that can be handled by set-theoretic and homologic
tools that are developed, and even these are not completely treated. Problems concerning homotopy, mappings of manifolds, applications to the study of group manifolds, etc., are all awaiting attention. But I hope that what is done here will serve as a useful basis for an attack on such problems.

The delay in publishing has been due to several factors. Since my delivery of the Colloquium Lectures on "Topology of Manifolds" at Vassar in September, 1942, in which the general outlines of this work were presented, the major part of a war has been fought, and a teacher in American universities need not be told what the attending demands, and the heavy post-war university enrollment of veterans, have done to the time that can be devoted to research. Also, most of the results in the later chapters, published here for the first time, were worked out with the euclidean $n$-space as locale. Resetting these in the generalized manifolds required not only revamping of proofs but taking advantage of the parallel advances in algebraic topology. New and more powerful tools were developing, such as the theory of cohomology and chain products, whose incorporation necessitated much revision but which justified themselves by the greater simplicity made possible in proofs. In many cases, proofs involving homology which were long and difficult became much simplified through the device of reverting to cohomology.

It also became apparent that the work would have to be topologically self-containing; the reader could not be expected to have previously read works on point set theory, topology of polyhedrals (combinatorial topology) and the newer algebraic topology. On the other hand, it was not possible to write a complete exposition of all these aspects of topology. The plan finally adopted was to develop the program from its simplest elements to its more complicated stages while simultaneously introducing the tools needed. Starting at first with general spaces, sufficient topological properties are introduced to characterize the basic 1-dimensional configurations (arc, 1-sphere). As a consequence, Chapter I is quite elementary. Some of the Schoenflies results in two dimensions are then given as well as some of the more modern plane point set theory—partly to furnish a natural basis and motive for the $n$-dimensional case and partly to present a unified treatment which takes advantage of the newer methods.

Algebraic topology is not introduced until needed—some topology of polyhedrals enters incidental to the material on the euclidean $n$-sphere in Chapter II, the more recent algebraic topology not being introduced until Chapter V. Although the treatment of these topics obviously could not be made in such general and complete fashion as in the companion volume by Lefschetz [L] in this series, enough is given to carry through the later chapters. The discerning reader will see many algebraic problems to be solved. Throughout the later chapters only an algebraic field is used as coefficient group, since, for example, the geometric form of the Alexander-Pontrjagin duality forms an important tool (three coefficient groups are usually involved—one to define the manifold, and one each for the homology theory of a subset $M$ and for the complement of $M$). However, it is impossible to do more in a work of this size than to sketch in the general
picture; the author hopes that other writers will fill in some of the gaps and bring the picture into sharper focus.

In an Appendix, I have pointed out some unsolved problems. Some of these may have very simple solutions; others (as for instance 1.1) are probably quite difficult. Such well-known (and difficult) classical problems as the classification of manifolds, conditions under which the $S^2$ in $S^3$ bounds a 3-cell, etc., are omitted.

References to the bibliography are enclosed in brackets, those involving capital letters such as [V] or [Mo] referring to books on topology, and those involving only lower case letters such as [a], [c] referring to miscellanea, mainly journal articles. Page numbers, etc., may be included, as in [a; 20] referring to page 20 of the article cited. Cross-references to items in the text are generally made by citing chapter and section; thus "V 12.2" refers to Chapter V, section 12.2. When a section number alone occurs, such as "12.2", the reference is to the chapter in which the citation occurs. References to formulae are enclosed in parentheses.

Along with the index of terms, there is included for easy reference an index of symbols. Certain symbols which refer to analogous concepts might easily be confused. The latter remark applies particularly to the symbols for homology and cohomology groups. The problem of symbolizing the various types of these groups which are encountered in the present work, and the corresponding Betti numbers, proved a serious one, and it is questionable if it has been satisfactorily solved!

I am grateful to those who have lent their advice, read some of the chapters or assisted in reading proofs; particularly to Professors Miriam C. Ayer, E. G. Begle, S. Kaplan, P. A. White and Gail S. Young; also to Dr. K. E. Butcher, Dr. E. H. Larguier and Messrs. M. L. Curtis and L. F. Hsieh. Aid in preparation of the manuscript was received from the Alexander Ziwet Fund, administered by the Executive Board of the Rackham School of Graduate Studies of the University of Michigan.

I wish to thank the American Mathematical Society for the honor and privilege of publishing this volume in its Colloquium series.

Ann Arbor, Michigan
December, 1948
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INTRODUCTION TO THE 1963 EDITION

This edition represents primarily a reprinting of the original book published in 1949; however, there have been some corrections made in the text and a list of errata has been added at the end of the book. In addition the NOTES which follow this Introduction have been added to this edition. For calling errors to my attention, as well as for assistance with the Notes, I am indebted to both colleagues and former students.

NOTES TO THE 1963 EDITION

Page 193; 7.2 Theorem. The "\((P, Q)_{*,1}\)" condition may be replaced by the weaker condition "\((P, Q, \sim)_{*,1}\)" defined on page 327. For a much simpler proof of this theorem see Theorem VI.2 on page 227 of my paper A certain class of topological properties, Bull. Amer. Math. Soc., vol. 66 (1960), pp. 205–239.


Page 257. After the proof of 5.8 Lemma, insert:

"As a consequence of Theorem V 18.31, Theorem 1.1, and Lemmas 5.6 and 5.8, we can show

5.8a Lemma. With \(P\) and \(Q\) as before,

\[ H_{*,1}(S; Q, 0; P, 0) = h^{*-1}(S; Q, P). \]

(This Lemma is needed in 7.2, for instance)"
Page 316; 1.1 Theorem. This theorem is valid for any orientable n-gem (See my paper A certain class of topological properties, loc. cit., especially Theorem II.5 thereof and the “Remark” following it.) A similar observation holds with regard to the following items in Chapter XI:

Page 319; 1.4 Theorem and 2.2 Lemma
Page 319; 1.5 Theorem
Page 320; 2.1 Theorem
Page 321; 2.3 Theorem
Page 325; 2.19 Theorem (although Theorem V.1 of the paper cited above is more general)
Page 326; 2.20 Corollary
Page 326; 2.21 Corollary (this holds for any orientable closed locally euclidean n-manifold S such that \( p_1(S) = 0 \), and M need be only 0-le and have property \((P, Q, \sim)^{-1}\). See Corollary V.1 of the paper cited above)
Page 326; 2.22 Theorem and Corollaries (see Theorem V.2 of the paper cited above)
Page 329; 3.5 Theorem (see Theorem II.4 of the paper cited above)
Page 339; 5.12 Theorem (valid for D any domain such that \( p^{-1}(D) \) is finite, in an orientable n-gem; a like remark holds for Corollary 5.13)
Page 340; 5.15 Theorem
Page 340; 5.16 Theorem
Page 343; 5.26 Theorem
Page 344; all Corollaries 5.27, 5.28, 5.29 and 5.31
Page 345; 6.5 Theorem.

Pages 327, 328; replace 3.3 Theorem and 3.4 Theorem, respectively, by Theorems II.1 and II.2 of the paper cited above.

Page 366; 2.14 Theorem. The weaker condition ‘‘\((P, Q, \sim)\)’’ may be substituted for the le’’ condition in the hypothesis.

Page 381; Problems 1.2 and 2.3. See M. Lubafski, An example of an absolute neighborhood retract, which is the common boundary of three regions in the 3-dimensional euclidean space, Fund. Math., vol. 40 (1953), pp. 29–38. (Notice also footnote 2), ibid., concerning an unpublished result of Gruba in 1937.)

Page 382; Problems 3.1 and 3.2. For solutions for certain types of "homology $n$-manifolds" over the reals mod 1 or a principal ideal ring, see, respectively, C. T. Yang, Transformation groups on a homological manifold, Trans. Amer. Math. Soc., vol. 87 (1958), pp. 261-283; and F. A. Raymond, Poincaré duality in homology manifolds, Dissertation, University of Michigan, 1958.


ADDITIONAL NOTES FOR THE 1979 PRINTING


For further information concerning the history of generalized manifolds since the publication of the present work, the reader can be referred to the following sources:


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Only books that are of special significance for the text and papers that are specifically referred to therein are included in this bibliography. For papers on manifolds of the classical type, not referred to herein, the reader is referred to the bibliographies in the Colloquium volumes of Lefschetz and the book of Seifert and Threlfall cited below. For additional citations in the general literature of Topology, reference may be made to the works of Lefschetz and Seifert-Threlfall as well as to the bibliographies in the Colloquium volumes of R. L. Moore and G. T. Whyburn cited below.

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### CHAIN AND HOMOLOGY GROUPS

Chain groups are designated by the letter "C" followed by suitable symbols; for example, "C'(K)." Cycle groups and groups of bounding cycles are designated similarly by use of the letters "Z" and "B", respectively. Consequently in the table below only the symbols for the homology groups are given; to obtain the corresponding chain, cycle or bounding cycle groups (where they exist), replace "H" by "C," "Z," or "B." Thus the cycle group corresponding to "H'(K)" is "Z'(K)." Hence to look up a group, as "Z'(S; M, L; G)," instead look up "H'(S; M, L; G)" in the index below; the "Z" group desired will be found defined on the page cited.

Betti numbers are generally designated by "p." Thus, to look up the meaning of "p(S; M, L; A, B)," turn to the page designated for "H'(S; M, L; A, B)." Similar remarks hold for the cohomology case. However, some Betti and co-Betti numbers are listed below, particularly where special definitions are required or a letter different from "p" (such as "q") is used.

Individual cycles and cocycles are variously denoted in the text by the letters "\(z\)," "\(z\)'" and "\(z\)" with appropriate indices. Open point sets are denoted below by "\(P\)," "\(Q\)," "\(I\)," "\(V\)," closed point sets by "\(A\)," "\(B\)," "\(J\)," "\(M\)," "\(L\)," a single point by "\(x\)." Also, both below and in the text, the letter "K" denotes a complex, "S" a space, "F" an algebraic field, and "G" an abelian group.

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Denumerable. A set is said to be denumerable if there exists a (1-1)-correspondence between its elements and the natural numbers $1, 2, \ldots, n, \ldots$. In the terminology of cardinal numbers, a set is denumerable if the cardinal number of its elements is $\aleph_0$. (See Countable.)

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ERRATA

In each instance, the first number refers to the page. “Line –n” refers to the
nth line from the bottom of the page.

62 Line 23. Insert “of $|Z^{n-2}|$” after “($n-r$)-cells”
76 Line –21. Change “Lemma 2.4” to “Lemmas 2.3 and 2.4”
128 Line 17. Insert “as in 6.3” after “$K$”
158 Line 20. Insert “and C-cycle $z$” after “$H_v(S)$”
180 Line 15. Insert “$(\mathbb{R}^n)$” after second “$z$” as well as before “$P$”
191 Lines 5 and 7. After “all” insert “arbitrarily small”
206 Line 1. Before “then” insert “such that for some closed set $K$ containing $M,$
$\gamma' \sim 0 \mod K$”; and before “such” insert “and contained in $K$”
237 Line –10. Before “base” insert “interior (relative to the $xy$-plane) of the”
283 Line 23. Insert “small enough” before “neighborhood”
292 Line –13. The exponent of “$g$” should be “$n-r-1$”
303 Line 8. Insert “ulc” before “open”
304 6.1 Lemma. Insert “compact” before “space”
327 3.1 Definition; 3.2 Definition. Insert “and $\bar{Q}$ is compact” before comma.
368 Line 1. Before the second period insert “and $M_0 = \{(0, 0, z) | 0 < z \leq 1\}”
389 Between “Moore, R. L.” and “Poincaré, H.” insert
“MULLIKIN, A.
[a] Certain theorems relating to plane connected point sets, Transactions of
the American Mathematical Society, vol. 24 (1922), pp. 144–162”