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COEFFICIENT REGIONS FOR SCHLICHT FUNCTIONS

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AND
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WITH A CHAPTER ON

THE REGION OF VALUES OF THE DERIVATIVE OF A SCHLICHT FUNCTION

BY
ARTHUR GRAD

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PREFACE

Instead of investigating various isolated extremal problems in the theory of schlicht functions, the authors have concentrated their efforts during the last three years on the investigation of the family of extremal schlicht functions in the large and this monograph is a presentation of the results of this research. For the sake of completeness and readability it has been found desirable to include in some places work that has been published elsewhere by ourselves or others. As most of the material is new, we have tried to point out carefully the material which already exists in published form.

In the calculus of variations there are two classical approaches: (a) study of specific problems using local variations; (b) study of a whole class of extremal problems and the investigation of the structure of the class as a whole. Variational methods in conformal mapping have been developed systematically in the last few years, beginning with a paper by M. Schiffer in 1938. The various publications on this subject during the last ten years have been mainly concerned with results of type (a) whereas we have tried to develop in this monograph a systematic approach to results of type (b), but we do not believe that our approach is the only one.

Since the investigation of extremal problems in conformal mapping embraces a rather wide field of research, we have confined ourselves to extremal problems relating to a finite number of the coefficients in the Taylor expansion of a function which is regular and schlicht inside the unit circle. Results of type (b) then concern the study of the region of values of the first n coefficients considered as a point in multi-dimensional euclidean space. This problem is only one of a host of problems that can be formulated in the theory of schlicht functions, and indeed a much more general problem is mentioned in Chapter I. The authors have chosen to investigate the coefficient problem not only because of its classical interest but also because it seems likely that the methods developed in this special case can be extended to many other problems. Dr. A. Grad has added a chapter in which he investigates the region of possible values of the derivative of a schlicht function at a fixed point inside the unit circle and his solution provides another example of these methods. A somewhat different version of his work has already appeared in hektographed form.

We have tried to make this monograph self-contained to as large an extent as practicable, and for this reason we have tried to keep the proofs and phraseology on as elementary a level as possible. This has lengthened the proofs in only a few cases and altogether it has increased the total length only slightly. We feel that sufficient background for reading this monograph is provided by a knowledge which is comparable to that contained in standard books on the theory of functions.

October, 1948.

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We are greatly indebted to the Mathematical Sciences Division of the Office of Naval Research for financial assistance in writing this monograph, and especially for support in carrying out computations. In particular, we are indebted to Dr. Mina Rees, Director of the Mathematical Sciences Division, for the interest she has shown in this work.

We wish to express our gratitude to Princeton University and to the University of Wisconsin for providing funds to print the colored frontispiece plates. Also we wish to thank Stanford University for funds made available in 1946 to make preliminary computations of the tables in the Appendix.

Finally, we are indebted to Mrs. Dolly Crane for her expert editorial assistance, and to Mr. Roy Nakata for the preparation of the diagrams and for his help in setting formulas into the final draft of the manuscript.

September, 1950

A. C. Schaeffer and D. C. Spencer
University of Wisconsin and Princeton University

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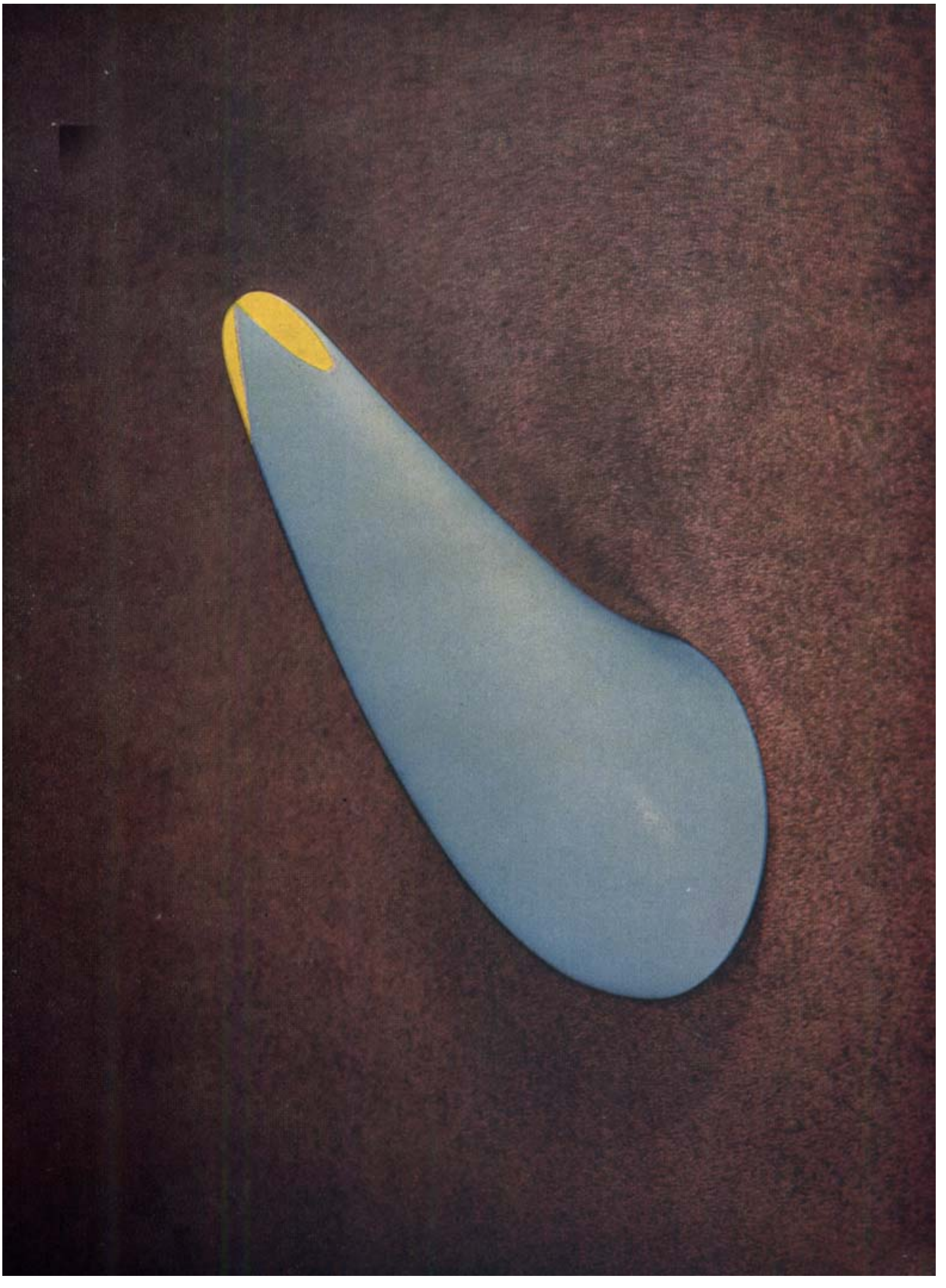

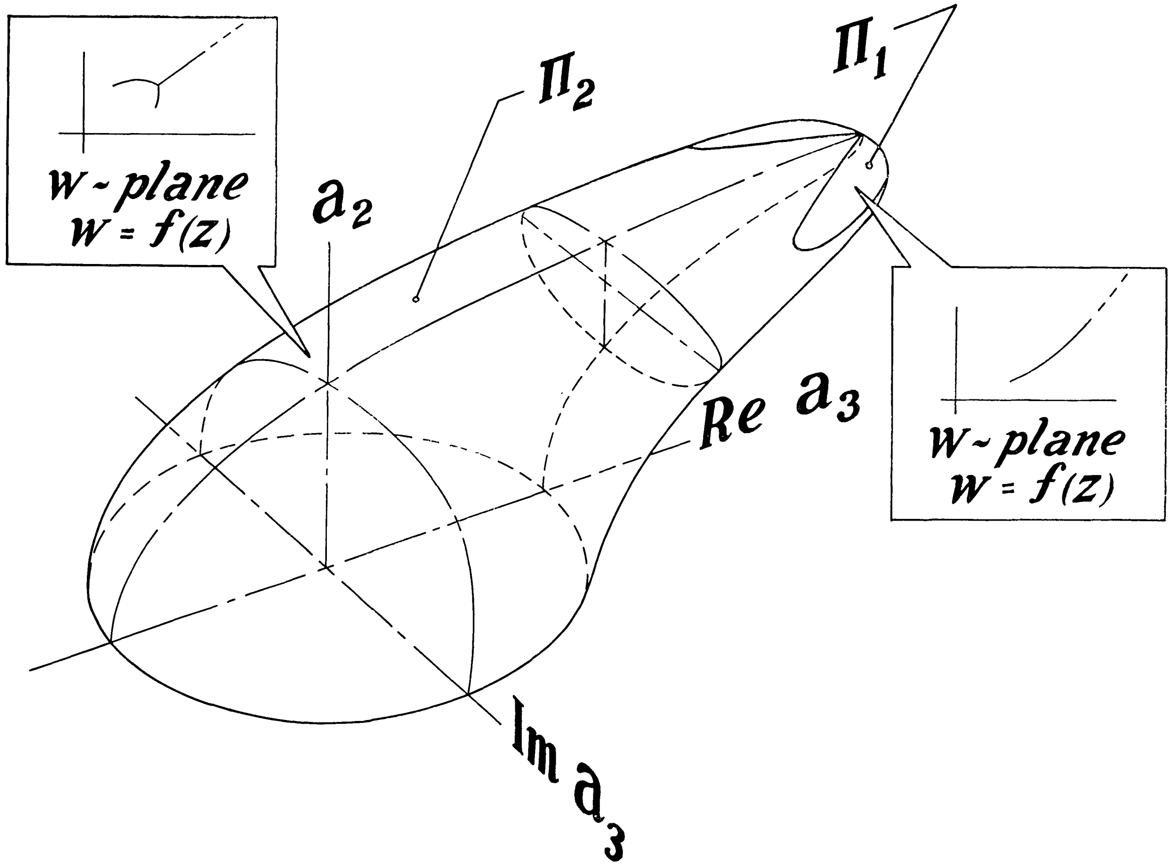

Plate I

The region of (a_2, a_3) when a_2 is real. (One half of solid shown.)

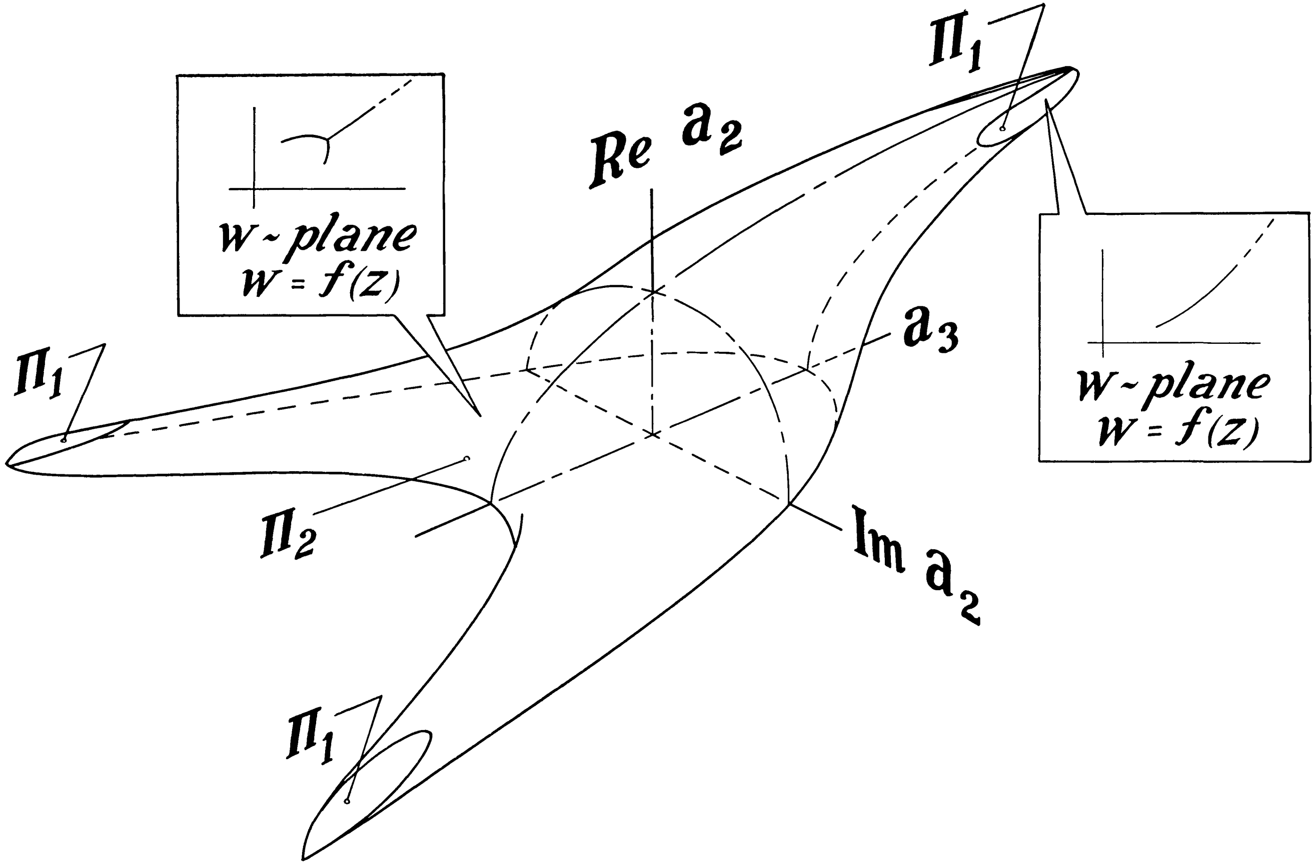

Plate II
The region of (a_2, a_3) when a_3 is real. (One half of solid shown.)

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APPENDIX

BOUNDARY OF V_3

TABLE I (a_2 REAL)

Part I (The portion Π_1 corresponding to unforked functions)

ρ	a_2	Re (a_3)	Im (a_3)	ρ	a_2	Re (a_3)	Im (a_3)
	$\varphi = 15°$				$\varphi = 60°$		
0.1	1.9994	2.9989	0.0278	0.1	1.9970	2.9938	0.0627
0.2	1.9984	2.9974	0.0454	0.2	1.9867	2.9718	0.1312
0.3	1.9975	2.9966	0.0544	0.3	1.9687	2.9340	0.1979
0.4	1.9971	2.9966	0.0565	0.4	1.9449	2.8870	0.2560
0.5	1.9971	2.9972	0.0532	0.5	1.9186	2.8395	0.3014
0.6	1.9974	2.9980	0.0460	0.6	1.8926	2.7982	0.3327
0.7	1.9979	2.9988	0.0363	0.7	1.8690	2.7658	0.3508
0.8	1.9984	2.9994	0.0258	0.8	1.8490	2.7424	0.3584
0.9	1.9987	2.9997	0.0163	0.9	1.8338	2.7274	0.3592
1.0	1.9989	2.9998	0.0111	1.0	1.8264	2.7208	0.3580
	$\varphi = 30°$				$\varphi = 67.5°$		
0.1	1.9980	2.9962	0.0514	0.1	1.9978	2.9954	0.0536
0.2	1.9938	2.9894	0.0891	0.2	1.9892	2.9760	0.1179
0.3	1.9895	2.9837	0.1132	0.3	1.9717	2.9354	0.1878
0.4	1.9862	2.9811	0.1250	0.4	1.9450	2.8745	0.2560
0.5	1.9841	2.9812	0.1266	0.5	1.9118	2.8017	0.3161
0.6	1.9832	2.9833	0.1205	0.6	1.8757	2.7283	0.3644
0.7	1.9831	2.9860	0.1090	0.7	1.8407	2.6628	0.3999
0.8	1.9833	2.9886	0.0948	0.8	1.8096	2.6101	0.4234
0.9	1.9834	2.9905	0.0810	0.9	1.7856	2.5728	0.4371
1.0	1.9834	2.9912	0.0732	1.0	1.7739	2.5556	0.4425
	$\varphi = 45°$				$\varphi = 75°$		
0.1	1.9968	2.9936	0.0654	0.1	1.9988	2.9975	0.0393
0.2	1.9882	2.9776	0.1235	0.2	1.9936	2.9852	0.0905
0.3	1.9768	2.9585	0.1693	0.3	1.9812	2.9539	0.1522
0.4	1.9651	2.9422	0.2010	0.4	1.9586	2.8954	0.2204
0.5	1.9546	2.9311	0.2189	0.5	1.9252	2.8089	0.2884
0.6	1.9460	2.9255	0.2248	0.6	1.8839	2.7045	0.3503
0.7	1.9393	2.9239	0.2212	0.7	1.8397	2.5972	0.4024
0.8	1.9342	2.9245	0.2115	0.8	1.7980	2.5009	0.4429
0.9	1.9304	2.9257	0.1998	0.9	1.7645	2.4268	0.4713
1.0	1.9284	2.9264	0.1925	1.0	1.7478	2.3909	0.4842

ρ	a_2	Re (a_3)	Im (a_3)
	$\varphi = 1.353000$ rad.		
0.1	1.9992	2.9981	0.0334
0.2	1.9953	2.9888	0.0780
0.3	1.9854	2.9636	0.1336
0.4	1.9663	2.9122	0.1977
0.5	1.9359	2.8291	0.2648
0.6	1.8957	2.7203	0.3285
0.7	1.8504	2.6012	0.3840
0.8	1.8062	2.4891	0.4287
0.9	1.7700	2.4000	0.4607
1.0	1.7518	2.3559	0.4755
	$\varphi = 1.400000$ rad.		
0.1	1.9995	2.9988	0.0268
0.2	1.9969	2.9926	0.0632
0.3	1.9901	2.9747	0.1101
0.4	1.9757	2.9348	0.1666
0.5	1.9507	2.8628	0.2294
0.6	1.9143	2.7575	0.2922
0.7	1.8702	2.6313	0.3492
0.8	1.8250	2.5048	0.3962
0.9	1.7867	2.3998	0.4304
1.0	1.7672	2.3465	0.4461
	$\varphi = 1.431000$ rad.		
0.1	1.9996	2.9992	0.0221
0.2	1.9978	2.9948	0.0526
0.3	1.9929	2.9818	0.0926
0.4	1.9822	2.9512	0.1422
0.5	1.9619	2.8913	0.1996
0.6	1.9301	2.7951	0.2596
0.7	1.8887	2.6704	0.3158
0.8	1.8443	2.5379	0.3629
0.9	1.8054	2.4238	0.3972
1.0	1.7853	2.3645	0.4129
	$\varphi = 1.466000$ rad.		
0.1	1.9998	2.9995	0.0168
0.2	1.9987	2.9970	0.0401
0.3	1.9958	2.9889	0.0711
0.4	1.9890	2.9695	0.1109
0.5	1.9752	2.9272	0.1592
0.6	1.9508	2.8501	0.2130
0.7	1.9153	2.7368	0.2660
0.8	1.8738	2.6046	0.3114
0.9	1.8356	2.4835	0.3445
1.0	1.8151	2.4183	0.3593

ρ	a_2	Re (a_3)	Im (a_3)
	$\varphi = 1.492000$ rad.		
0.1	1.9999	2.9997	0.0127
0.2	1.9993	2.9983	0.0304
0.3	1.9976	2.9936	0.0543
0.4	1.9935	2.9816	0.0854
0.5	1.9846	2.9539	0.1246
0.6	1.9672	2.8974	0.1707
0.7	1.9387	2.8027	0.2189
0.8	1.9017	2.6791	0.2617
0.9	1.8654	2.5576	0.2928
1.0	1.8452	2.4894	0.3064
	$\varphi = 1.518000$ rad.		
0.1	1.9999	2.9999	0.0085
0.2	1.9997	2.9992	0.0205
0.3	1.9989	2.9971	0.0367
0.4	1.9969	2.9913	0.0583
0.5	1.9925	2.9773	0.0862
0.6	1.9828	2.9452	0.1211
0.7	1.9641	2.8805	0.1610
0.8	1.9351	2.7788	0.1993
0.9	1.9028	2.6650	0.2275
1.0	1.8835	2.5963	0.2394
	$\varphi = 1.546000$ rad.		
0.1	2.0000	3.0000	0.0040
0.2	1.9999	2.9998	0.0097
0.3	1.9997	2.9993	0.0174
0.4	1.9993	2.9980	0.0277
0.5	1.9983	2.9947	0.0414
0.6	1.9957	2.9861	0.0595
0.7	1.9895	2.9640	0.0831
0.8	1.9756	2.9127	0.1107
0.9	1.9533	2.8294	0.1341
1.0	1.9368	2.7673	0.1437
	$\varphi = 1.560000$ rad.		
0.1	2.0000	3.0000	0.0017
0.2	2.0000	3.0000	0.0042
0.3	2.0000	2.9999	0.0076
0.4	1.9999	2.9996	0.0121
0.5	1.9997	2.9990	0.0181
0.6	1.9992	2.9973	0.0262
0.7	1.9978	2.9924	0.0374
0.8	1.9937	2.9770	0.0528
0.9	1.9824	2.9337	0.0703
1.0	1.9698	2.8847	0.0786

Part II (The portion **II**$_2$ corresponding to forked functions)

μ	a_2	Re (a_3)	Im (a_3)
φ = 0.523599 rad.			
0.0000000	1.9812	2.9936	0.0000
0.0931004	1.9834	2.9912	0.0732
φ = 0.785398 rad.			
0.0000000	1.9043	2.9334	0.0000
0.1500000	1.9102	2.9312	0.1010
0.3034926	1.9284	2.9264	0.1925
φ = 0.916000 rad.			
0.0000000	1.8220	2.8424	0.0000
0.1600000	1.8291	2.8413	0.1087
0.3200000	1.8499	2.8405	0.2037
0.4706602	1.8819	2.8460	0.2703
φ = 0.982000 rad.			
0.0000000	1.7640	2.7692	0.0000
0.2000000	1.7753	2.7681	0.1398
0.4000000	1.8088	2.7710	0.2531
0.5724916	1.8546	2.7888	0.3136
φ = 1.047198 rad.			
0.0000000	1.6931	2.6736	0.0000
0.1700000	1.7017	2.6726	0.1282
0.3400000	1.7269	2.6729	0.2391
0.5100000	1.7683	2.6842	0.3185
0.6848541	1.8264	2.7208	0.3580
φ = 1.090000 rad.			
0.0000000	1.6383	2.5963	0.0000
0.1500000	1.6451	2.5951	0.1205
0.3000000	1.6655	2.5937	0.2285
0.4500000	1.6989	2.5985	0.3131
0.6000000	1.7447	2.6186	0.3672
0.7650357	1.8081	2.6704	0.3871
φ = 1.132000 rad.			
0.0000000	1.5771	2.5081	0.0000
0.1600000	1.5852	2.5060	0.1374
0.3200000	1.6092	2.5028	0.2586
0.4800000	1.6485	2.5077	0.3504
0.6400000	1.7020	2.5328	0.4046
0.8486667	1.7909	2.6171	0.4147
φ = 1.174000 rad.			
0.0000000	1.5078	2.4067	0.0000
0.1900000	1.5197	2.4022	0.1753
0.3800000	1.5549	2.3959	0.3218
0.5700000	1.6119	2.4068	0.4186

μ	a_2	Re (a_3)	Im (a_3)
φ = 1.174000 rad.			
0.7500000	1.6840	2.4540	0.4560
0.9371865	1.7753	2.5611	0.4402
φ = 1.208000 rad.			
0.0000000	1.4451	2.3144	0.0000
0.1900000	1.4575	2.3080	0.1887
0.3800000	1.4942	2.2969	0.3461
0.5700000	1.5534	2.3022	0.4501
0.7600000	1.6327	2.3496	0.4921
1.0124021	1.7644	2.5151	0.4580
φ = 1.242000 rad.			
0.0000000	1.3758	2.2129	0.0000
0.1500000	1.3840	2.2070	0.1636
0.3000000	1.4082	2.1932	0.3094
0.4500000	1.4476	2.1818	0.4235
0.6000000	1.5010	2.1867	0.4982
0.7500000	1.5670	2.2217	0.5314
1.0907697	1.7558	2.4698	0.4722
φ = 1.275000 rad.			
0.0000000	1.3017	2.1054	0.0000
0.1200000	1.3072	2.1001	0.1439
0.2400000	1.3236	2.0861	0.2773
0.3600000	1.3505	2.0688	0.3913
0.4800000	1.3874	2.0557	0.4797
0.6000000	1.4333	2.0554	0.5398
0.7200000	1.4875	2.0756	0.5713
1.1698113	1.7501	2.4284	0.4813
φ = 1.308997 rad.			
0.0000000	1.2173	1.9857	0.0000
0.1800000	1.2305	1.9695	0.2350
0.3600000	1.2694	1.9329	0.4304
0.5400000	1.3317	1.9043	0.5611
0.7200000	1.4143	1.9163	0.6211
0.9000000	1.5139	1.9941	0.6183
1.0800000	1.6273	2.1531	0.5669
1.1700000	1.6884	2.2655	0.5274
1.2542651	1.7478	2.3909	0.4842
φ = 1.331000 rad.			
0.0000000	1.1579	1.9035	0.0000
0.1500000	1.1675	1.8892	0.2123
0.3000000	1.1961	1.8535	0.3979
0.4500000	1.2422	1.8139	0.5377
0.6000000	1.3041	1.7923	0.6242
0.7500000	1.3795	1.8082	0.6593
1.3105349	1.7488	2.3710	0.4818

μ	a_2	Re (a_3)	Im (a_3)
φ = 1.353000 rad.			
0.0000000	1.0943	1.8179	0.0000
0.2000000	1.1124	1.7872	0.3003
0.4000000	1.1651	1.7202	0.5323
0.6000000	1.2480	1.6724	0.6626
0.8000000	1.3555	1.6962	0.6961
1.0000000	1.4824	1.8228	0.6568
1.1000000	1.5516	1.9288	0.6180
1.2000000	1.6240	2.0640	0.5702
1.3680436	1.7518	2.3559	0.4755
φ = 1.377000 rad.			
0.0000000	1.0196	1.7211	0.0000
0.1500000	1.0306	1.6981	0.2516
0.3000000	1.0628	1.6399	0.4662
0.4500000	1.1145	1.5734	0.6202
0.6000000	1.1831	1.5285	0.7079
0.7500000	1.2658	1.5293	0.7366
0.9000000	1.3600	1.5903	0.7192
1.4321799	1.7581	2.3465	0.4634
φ = 1.400000 rad.			
0.0000000	0.9423	1.6255	0.0000
0.1900000	0.9613	1.5789	0.3453
0.3700000	1.0124	1.4798	0.5939
0.5600000	1.0962	1.3869	0.7387
0.7500000	1.2044	1.3676	0.7758
0.9300000	1.3240	1.4465	0.7415
1.0200000	1.3887	1.5245	0.7077
1.1200000	1.4637	1.6412	0.6615
1.2200000	1.5416	1.7888	0.6087
1.3100000	1.6137	1.9474	0.5576
1.4949916	1.7672	2.3465	0.4461
φ = 1.422000 rad.			
0.0000000	0.8625	1.5321	0.0000
0.1200000	0.8708	1.5068	0.2494
0.2400000	0.8952	1.4396	0.4681
0.3600000	0.9346	1.3519	0.6355
0.4800000	0.9870	1.2687	0.7449
0.6000000	1.0506	1.2112	0.8009
0.7200000	1.1235	1.1930	0.8137
0.8400000	1.2039	1.2213	0.7945
0.9600000	1.2905	1.2983	0.7534
1.0800000	1.3821	1.4236	0.6979
1.2000000	1.4778	1.5953	0.6337
1.5562835	1.7793	2.3568	0.4238
φ = 1.444000 rad.			
0.0000000	0.7759	1.4378	0.0000

μ	a_2	Re (a_3)	Im (a_3)
φ = 1.444000 rad.			
0.1300000	0.7867	1.3983	0.3035
0.2600000	0.8183	1.2970	0.5560
0.3900000	0.8684	1.1736	0.7296
0.5200000	0.9340	1.0685	0.8230
0.6500000	1.0122	1.0091	0.8516
0.7800000	1.1002	1.0088	0.8347
0.9100000	1.1959	1.0708	0.7889
1.0400000	1.2975	1.1934	0.7262
1.1700000	1.4039	1.3731	0.6548
1.3000000	1.5139	1.6059	0.5795
1.6187370	1.7951	2.3797	0.3951
φ = 1.466000 rad.			
0.0000000	0.6815	1.3438	0.0000
0.1000000	0.6888	1.3107	0.2726
0.1900000	0.7075	1.2325	0.4889
0.2800000	0.7368	1.1270	0.6598
0.3700000	0.7755	1.0154	0.7796
0.4600000	0.8222	0.9157	0.8515
0.5600000	0.8821	0.8336	0.8853
0.6500000	0.9418	0.7927	0.8848
0.7500000	1.0134	0.7868	0.8603
0.8400000	1.0817	0.8169	0.8242
0.9400000	1.1611	0.8878	0.7746
1.0400000	1.2434	0.9956	0.7194
1.1200000	1.3111	1.1067	0.6733
1.1800000	1.3627	1.2036	0.6383
1.2400000	1.4149	1.3117	0.6033
1.3100000	1.4767	1.4514	0.5627
1.3700000	1.5302	1.5823	0.5284
1.4400000	1.5931	1.7476	0.4890
1.5000000	1.6476	1.8997	0.4559
1.5450000	1.6886	2.0199	0.4315
1.5900000	1.7299	2.1453	0.4074
1.6350000	1.7714	2.2758	0.3837
1.6823271	1.8151	2.4183	0.3593
φ = 1.483000 rad.			
0.0000000	0.6022	1.2724	0.0000
0.1400000	0.6183	1.1879	0.4233
0.2800000	0.6641	0.9899	0.7273
0.4200000	0.7342	0.7843	0.8792
0.5600000	0.8224	0.6443	0.9145
0.7000000	0.9234	0.5954	0.8812
0.8400000	1.0336	0.6367	0.8143
0.9800000	1.1502	0.7587	0.7340
1.1200000	1.2716	0.9515	0.6506
1.2600000	1.3965	1.2069	0.5691
1.4000000	1.5240	1.5188	0.4916
1.7322273	1.8339	2.4613	0.3261

$\varphi = 1.500000$ rad.

μ	a_2	Re (a_3)	Im (a_3)
0.0000000	0.5162	1.2034	0.0000
0.1400000	0.5349	1.0846	0.4905
0.2800000	0.5873	0.8226	0.8070
0.4200000	0.6655	0.5753	0.9295
0.5600000	0.7616	0.4250	0.9274
0.7000000	0.8698	0.3850	0.8659
0.8400000	0.9859	0.4428	0.7822
0.9800000	1.1076	0.5826	0.6943
1.1200000	1.2332	0.7918	0.6095
1.2600000	1.3616	1.0614	0.5304
1.4000000	1.4921	1.3852	0.4576
1.7827784	1.8560	2.5180	0.2874

$\varphi = 1.518000$ rad.

μ	a_2	Re (a_3)	Im (a_3)
0.0000000	0.4160	1.1347	0.0000
0.0800000	0.4237	1.0694	0.3645
0.1600000	0.4457	0.9012	0.6569
0.2400000	0.4803	0.6902	0.8452
0.3200000	0.5249	0.4896	0.9381
0.4000000	0.5771	0.3287	0.9626
0.4800000	0.6352	0.2167	0.9447
0.5600000	0.6976	0.1524	0.9036
0.6400000	0.7633	0.1306	0.8515
0.7200000	0.8316	0.1454	0.7952
0.7800000	0.8840	0.1774	0.7528
0.8400000	0.9374	0.2254	0.7112
0.9100000	1.0006	0.2994	0.6644
0.9700000	1.0555	0.3770	0.6262
1.0400000	1.1201	0.4826	0.5838
1.1000000	1.1760	0.5853	0.5493
1.1600000	1.2323	0.6984	0.5167
1.2200000	1.2890	0.8215	0.4856
1.2800000	1.3459	0.9542	0.4561
1.4700000	1.5277	1.4336	0.3718
1.6500000	1.7016	1.9663	0.3026
1.8369980	1.8835	2.5963	0.2394

$\varphi = 1.531000$ rad.

μ	a_2	Re (a_3)	Im (a_3)
0.0000000	0.3361	1.0894	0.0000
0.1000000	0.3507	0.9363	0.5404
0.2000000	0.3911	0.6048	0.8657
0.3000000	0.4505	0.2898	0.9728
0.4000000	0.5225	0.0735	0.9563
0.5000000	0.6025	−0.0424	0.8892
0.6000000	0.6877	−0.0777	0.8077
0.7000000	0.7765	−0.0510	0.7272
0.8000000	0.8677	0.0241	0.6528
0.9000000	0.9607	0.1387	0.5860
1.0000000	1.0550	0.2867	0.5264
1.1000000	1.1502	0.4644	0.4732
1.2000000	1.2462	0.6690	0.4256
1.3000000	1.3428	0.8988	0.3828
1.4000000	1.4398	1.1524	0.3441
1.8765924	1.9065	2.6670	0.1989

$\varphi = 1.546000$ rad.

μ	a_2	Re (a_3)	Im (a_3)
0.0000000	0.2329	1.0439	0.0000
0.0400000	0.2363	0.9882	0.3321
0.0800000	0.2463	0.8390	0.6112
0.1200000	0.2620	0.6383	0.8093
0.1600000	0.2826	0.4276	0.9266
0.2000000	0.3070	0.2341	0.9798
0.2400000	0.3344	0.0703	0.9889
0.2800000	0.3642	−0.0612	0.9706
0.3200000	0.3958	−0.1626	0.9369
0.3600000	0.4288	−0.2380	0.8954
0.4000000	0.4629	−0.2915	0.8509
0.4400000	0.4978	−0.3266	0.8062
0.5100000	0.5607	−0.3528	0.7314
0.5800000	0.6250	−0.3440	0.6637
0.6400000	0.6811	−0.3147	0.6118
0.7000000	0.7377	−0.2689	0.5652
0.7700000	0.8045	−0.1978	0.5168
0.8300000	0.8621	−0.1234	0.4798
0.8900000	0.9200	−0.0381	0.4463
0.9600000	0.9879	0.0744	0.4112
1.1600000	1.1832	0.4647	0.3290
1.3500000	1.3699	0.9219	0.2686
1.5400000	1.5575	1.4579	0.2197
1.7300000	1.7456	2.0700	0.1790
1.9227228	1.9368	2.7673	0.1437

$\varphi = 1.553000$ rad.

μ	a_2	Re (a_3)	Im (a_3)
0.0000000	0.1790	1.0263	0.0000
0.3000000	0.3493	−0.3596	0.8699
0.4000000	0.4382	−0.4811	0.7319
0.5000000	0.5311	−0.4977	0.6172
0.6000000	0.6261	−0.4514	0.5268
0.7000000	0.7225	−0.3622	0.4554
0.8000000	0.8198	−0.2396	0.3979
0.9000000	0.9176	−0.0888	0.3508
1.0000000	1.0159	0.0871	0.3115
1.9444109	1.9526	2.8226	0.1135

μ	a_2	Re (a_3)	Im (a_3)
	φ = 1.560000 rad.		
0.0000000	0.1194	1.0119	0.0000
0.0200000	0.1210	0.9577	0.3256
0.0400000	0.1259	0.8116	0.6018
0.0600000	0.1336	0.6121	0.8014
0.0800000	0.1437	0.3983	0.9234
0.1000000	0.1557	0.1970	0.9826
0.1200000	0.1693	0.0208	0.9976
0.1400000	0.1840	−0.1268	0.9846
0.1600000	0.1996	−0.2476	0.9554
0.1800000	0.2160	−0.3451	0.9175
0.2000000	0.2329	−0.4231	0.8761
0.2200000	0.2503	−0.4851	0.8338
0.2700000	0.2952	−0.5886	0.7340
0.3300000	0.3509	−0.6482	0.6328
0.3900000	0.4079	−0.6651	0.5514
0.4500000	0.4656	−0.6545	0.4861
0.5200000	0.5335	−0.6180	0.4250
0.5900000	0.6020	−0.5618	0.3761
0.7900000	0.7990	−0.3198	0.2785
0.9800000	0.9872	0.0011	0.2190
1.1800000	1.1860	0.4241	0.1748
1.3800000	1.3852	0.9307	0.1420
1.5700000	1.5745	1.4878	0.1173
1.7700000	1.7740	2.1534	0.0961
1.9661996	1.9698	2.8847	0.0786
	φ = 1.564000 rad.		
0.0000000	0.0814	1.0056	0.0000
0.1650000	0.1840	−0.5755	0.7916
0.2200000	0.2346	−0.7051	0.6482
0.2750000	0.2868	−0.7577	0.5408
0.3300000	0.3399	−0.7708	0.4608
0.3850000	0.3935	−0.7607	0.3997
0.4400000	0.4475	−0.7347	0.3520
0.4950000	0.5017	−0.6968	0.3137
0.5500000	0.5560	−0.6492	0.2823
0.6050000	0.6105	−0.5929	0.2562
1.9786950	1.9804	2.9240	0.0553
	φ = 1.567000 rad.		
0.0000000	0.0499	1.0021	0.0000
0.1290000	0.1383	−0.7208	0.6721
0.1720000	0.1791	−0.8130	0.5340

μ	a_2	Re (a_3)	Im (a_3)
	φ = 1.567000 rad.		
0.2150000	0.2207	−0.8494	0.4389
0.2580000	0.2628	−0.8592	0.3710
0.3010000	0.3051	−0.8538	0.3205
0.3440000	0.3476	−0.8383	0.2816
0.3870000	0.3902	−0.8154	0.2508
0.4300000	0.4329	−0.7864	0.2258
0.4730000	0.4756	−0.7522	0.2051
1.9880879	1.9887	2.9558	0.0351
	φ = 1.570000 rad.		
0.0000000	0.0130	1.0001	0.0000
0.0020000	0.0131	0.9536	0.3015
0.0040000	0.0136	0.8261	0.5637
0.0060000	0.0143	0.6470	0.7626
0.0080000	0.0152	0.4482	0.8940
0.0100000	0.0164	0.2537	0.9673
0.0120000	0.0177	0.0769	0.9970
0.0140000	0.0191	−0.0769	0.9970
0.0160000	0.0206	−0.2075	0.9781
0.0180000	0.0222	−0.3169	0.9483
0.0200000	0.0238	−0.4082	0.9126
0.0220000	0.0255	−0.4843	0.8745
0.0260000	0.0290	−0.6013	0.7984
0.0300000	0.0327	−0.6845	0.7280
0.0340000	0.0364	−0.7451	0.6655
0.0380000	0.0401	−0.7901	0.6109
0.0490000	0.0507	−0.8667	0.4942
0.0600000	0.0614	−0.9071	0.4126
0.0800000	0.0810	−0.9423	0.3155
0.1000000	0.1008	−0.9568	0.2547
0.1500000	0.1506	−0.9625	0.1712
0.2000000	0.2004	−0.9515	0.1287
0.3000000	0.3003	−0.9061	0.0857
0.4000000	0.4002	−0.8378	0.0641
0.6000000	0.6001	−0.6389	0.0422
0.8000000	0.8001	−0.3593	0.0311
1.0000000	1.0001	0.0005	0.0243
1.2000000	1.2001	0.4404	0.0197
1.4000000	1.4001	0.9603	0.0163
1.6000000	1.6001	1.5603	0.0136
1.8000000	1.8000	2.2402	0.0115
1.9974989	1.9975	2.9902	0.0098

TABLE II (a_3 REAL)

Part I (The portion Π_1 corresponding to unforked functions)

ρ	Re (a_2)	Im (a_2)	a_3	ρ	Re (a_2)	Im (a_2)	a_3
	$\varphi = 15°$				$\varphi = 60°$		
0.1	1.9994	0.0093	2.9991	0.1	1.9969	0.0209	2.9945
0.2	1.9983	0.0151	2.9977	0.2	1.9862	0.0438	2.9747
0.3	1.9975	0.0181	2.9970	0.3	1.9675	0.0663	2.9407
0.4	1.9970	0.0188	2.9971	0.4	1.9430	0.0860	2.8983
0.5	1.9970	0.0177	2.9976	0.5	1.9159	0.1014	2.8555
0.6	1.9974	0.0153	2.9984	0.6	1.8893	0.1119	2.8179
0.7	1.9979	0.0121	2.9990	0.7	1.8653	0.1178	2.7879
0.8	1.9984	0.0086	2.9995	0.8	1.8451	0.1201	2.7658
0.9	1.9987	0.0054	2.9998	0.9	1.8299	0.1200	2.7509
1.0	1.9989	0.0037	2.9999	1.0	1.8225	0.1194	2.7443
	$\varphi = 30°$				$\varphi = 67.5°$		
0.1	1.9979	0.0171	2.9967	0.1	1.9978	0.0179	2.9958
0.2	1.9936	0.0297	2.9907	0.2	1.9888	0.0394	2.9783
0.3	1.9891	0.0377	2.9859	0.3	1.9707	0.0630	2.9414
0.4	1.9857	0.0416	2.9837	0.4	1.9431	0.0864	2.8859
0.5	1.9837	0.0421	2.9839	0.5	1.9087	0.1074	2.8195
0.6	1.9828	0.0400	2.9857	0.6	1.8716	0.1245	2.7525
0.7	1.9827	0.0362	2.9880	0.7	1.8356	0.1371	2.6926
0.8	1.9830	0.0314	2.9901	0.8	1.8038	0.1454	2.6442
0.9	1.9833	0.0269	2.9916	0.9	1.7793	0.1501	2.6096
1.0	1.9832	0.0243	2.9921	1.0	1.7674	0.1519	2.5936
	$\varphi = 45°$				$\varphi = 75°$		
0.1	1.9967	0.0218	2.9943	0.1	1.9988	0.0131	2.9977
0.2	1.9878	0.0412	2.9802	0.2	1.9934	0.0302	2.9865
0.3	1.9760	0.0565	2.9634	0.3	1.9805	0.0510	2.9578
0.4	1.9640	0.0670	2.9490	0.4	1.9572	0.0744	2.9037
0.5	1.9532	0.0728	2.9393	0.5	1.9227	0.0985	2.8236
0.6	1.9446	0.0746	2.9341	0.6	1.8800	0.1213	2.7271
0.7	1.9379	0.0732	2.9323	0.7	1.8343	0.1412	2.6282
0.8	1.9329	0.0698	2.9322	0.8	1.7911	0.1574	2.5398
0.9	1.9292	0.0658	2.9325	0.9	1.7563	0.1690	2.4722
1.0	1.9273	0.0633	2.9327	1.0	1.7391	0.1743	2.4395

ρ	Re (a_2)	Im (a_2)	a_3
	$\varphi = 1.353000$ rad.		
0.1	1.9991	0.0111	2.9983
0.2	1.9951	0.0260	2.9898
0.3	1.9849	0.0447	2.9666
0.4	1.9652	0.0666	2.9189
0.5	1.9338	0.0903	2.8415
0.6	1.8922	0.1138	2.7401
0.7	1.8454	0.1355	2.6294
0.8	1.7997	0.1538	2.5257
0.9	1.7620	0.1676	2.4438
1.0	1.7432	0.1741	2.4034
	$\varphi = 1.400000$ rad.		
0.1	1.9994	0.0089	2.9989
0.2	1.9968	0.0211	2.9932
0.3	1.9897	0.0368	2.9767
0.4	1.9750	0.0560	2.9395
0.5	1.9491	0.0780	2.8720
0.6	1.9116	0.1010	2.7729
0.7	1.8661	0.1233	2.6544
0.8	1.8194	0.1430	2.5360
0.9	1.7797	0.1583	2.4381
1.0	1.7594	0.1658	2.3885
	$\varphi = 1.431000$ rad.		
0.1	1.9996	0.0074	2.9993
0.2	1.9978	0.0176	2.9953
0.3	1.9927	0.0309	2.9833
0.4	1.9816	0.0477	2.9547
0.5	1.9608	0.0676	2.8982
0.6	1.9280	0.0893	2.8072
0.7	1.8855	0.1111	2.6890
0.8	1.8396	0.1308	2.5637
0.9	1.7995	0.1465	2.4561
1.0	1.7786	0.1541	2.4002
	$\varphi = 1.466000$ rad.		
0.1	1.9998	0.0056	2.9996
0.2	1.9987	0.0134	2.9972
0.3	1.9956	0.0237	2.9898
0.4	1.9887	0.0371	2.9716
0.5	1.9745	0.0537	2.9315
0.6	1.9494	0.0727	2.8581
0.7	1.9130	0.0927	2.7497
0.8	1.8705	0.1114	2.6231
0.9	1.8312	0.1264	2.5073
1.0	1.8102	0.1337	2.4449

ρ	Re (a_2)	Im (a_2)	a_3
	$\varphi = 1.492000$ rad.		
0.1	1.9999	0.0042	2.9998
0.2	1.9993	0.0101	2.9984
0.3	1.9975	0.0181	2.9941
0.4	1.9933	0.0285	2.9828
0.5	1.9841	0.0418	2.9565
0.6	1.9663	0.0579	2.9025
0.7	1.9372	0.0755	2.8112
0.8	1.8995	0.0926	2.6919
0.9	1.8624	0.1063	2.5743
1.0	1.8417	0.1129	2.5082
	$\varphi = 1.518000$ rad.		
0.1	1.9999	0.0028	2.9999
0.2	1.9997	0.0068	2.9993
0.3	1.9988	0.0123	2.9973
0.4	1.9968	0.0194	2.9919
0.5	1.9923	0.0288	2.9786
0.6	1.9824	0.0407	2.9476
0.7	1.9633	0.0548	2.8850
0.8	1.9339	0.0692	2.7860
0.9	1.9011	0.0810	2.6747
1.0	1.8815	0.0865	2.6073
	$\varphi = 1.546000$ rad.		
0.1	2.0000	0.0013	3.0000
0.2	1.9999	0.0032	2.9998
0.3	1.9997	0.0058	2.9994
0.4	1.9993	0.0092	2.9982
0.5	1.9982	0.0138	2.9949
0.6	1.9956	0.0199	2.9867
0.7	1.9893	0.0279	2.9652
0.8	1.9752	0.0375	2.9148
0.9	1.9527	0.0462	2.8326
1.0	1.9361	0.0502	2.7710
	$\varphi = 1.560000$ rad.		
0.1	2.0000	0.0006	3.0000
0.2	2.0000	0.0014	3.0000
0.3	2.0000	0.0025	2.9999
0.4	1.9999	0.0040	2.9996
0.5	1.9997	0.0060	2.9990
0.6	1.9991	0.0087	2.9974
0.7	1.9978	0.0125	2.9926
0.8	1.9936	0.0177	2.9775
0.9	1.9823	0.0237	2.9345
1.0	1.9696	0.0268	2.8858

Part II (The portion Π_2 corresponding to forked functions)

μ	Re (a_2)	Im (a_2)	a_3
$\varphi = 0.523599$ rad.			
0.0000000	1.9812	0.0000	2.9936
0.0931004	1.9832	0.0243	2.9921
$\varphi = 0.785398$ rad.			
0.0000000	1.9043	0.0000	2.9334
0.1500000	1.9100	0.0329	2.9329
0.3034926	1.9273	0.0633	2.9327
$\varphi = 0.916000$ rad.			
0.0000000	1.8220	0.0000	2.8424
0.1600000	1.8287	0.0350	2.8434
0.3200000	1.8488	0.0662	2.8478
0.4706602	1.8797	0.0891	2.8588
$\varphi = 0.982000$ rad.			
0.0000000	1.7640	0.0000	2.7692
0.2000000	1.7747	0.0448	2.7717
0.4000000	1.8069	0.0824	2.7826
0.5724916	1.8516	0.1038	2.8064
$\varphi = 1.047198$ rad.			
0.0000000	1.6931	0.0000	2.6736
0.1700000	1.7012	0.0408	2.6756
0.3400000	1.7252	0.0770	2.6836
0.5100000	1.7652	0.1044	2.7030
0.6848541	1.8225	0.1194	2.7443
$\varphi = 1.090000$ rad.			
0.0000000	1.6383	0.0000	2.5963
0.1500000	1.6447	0.0382	2.5979
0.3000000	1.6639	0.0731	2.6037
0.4500000	1.6959	0.1018	2.6173
0.6000000	1.7404	0.1214	2.6442
0.7650357	1.8034	0.1300	2.6983
$\varphi = 1.132000$ rad.			
0.0000000	1.5771	0.0000	2.5081
0.1600000	1.5846	0.0434	2.5097
0.3200000	1.6071	0.0828	2.5162
0.4800000	1.6445	0.1143	2.5321
0.6400000	1.6966	0.1347	2.5649
0.8486667	1.7854	0.1406	2.6497

μ	Re (a_2)	Im (a_2)	a_3
$\varphi = 1.174000$ rad.			
0.0000000	1.5078	0.0000	2.4067
0.1900000	1.5187	0.0554	2.4086
0.3800000	1.5515	0.1037	2.4175
0.5700000	1.6059	0.1386	2.4429
0.7500000	1.6769	0.1545	2.4960
0.9371865	1.7689	0.1509	2.5987
$\varphi = 1.208000$ rad.			
0.0000000	1.4451	0.0000	2.3144
0.1900000	1.4563	0.0594	2.3157
0.3800000	1.4900	0.1116	2.3228
0.5700000	1.5462	0.1497	2.3458
0.7600000	1.6240	0.1682	2.4005
1.0124021	1.7573	0.1587	2.5565
$\varphi = 1.242000$ rad.			
0.0000000	1.3758	0.0000	2.2129
0.1500000	1.3830	0.0512	2.2131
0.3000000	1.4047	0.0986	2.2149
0.4500000	1.4409	0.1386	2.2226
0.6000000	1.4916	0.1678	2.2427
0.7500000	1.5562	0.1835	2.2844
1.0907697	1.7479	0.1656	2.5146
$\varphi = 1.275000$ rad.			
0.0000000	1.3017	0.0000	2.1054
0.1200000	1.3064	0.0447	2.1050
0.2400000	1.3207	0.0874	2.1045
0.3600000	1.3446	0.1260	2.1054
0.4800000	1.3783	0.1587	2.1110
0.6000000	1.4215	0.1835	2.1251
0.7200000	1.4741	0.1992	2.1528
1.1698113	1.7417	0.1709	2.4757
$\varphi = 1.308997$ rad.			
0.0000000	1.2173	0.0000	1.9857
0.1800000	1.2284	0.0730	1.9835
0.3600000	1.2618	0.1388	1.9802
0.5400000	1.3180	0.1901	1.9853
0.7200000	1.3970	0.2207	2.0144
0.9000000	1.4968	0.2267	2.0877
1.0800000	1.6139	0.2089	2.2265
1.1700000	1.6774	0.1927	2.3261
1.2542651	1.7391	0.1743	2.4395

$\varphi = 1.331000$ rad.

μ	Re (a_2)	Im (a_2)	a_3
0.0000000	1.1579	0.0000	1.9035
0.1500000	1.1657	0.0653	1.9011
0.3000000	1.1894	0.1262	1.8957
0.4500000	1.2294	0.1784	1.8919
0.6000000	1.2858	0.2175	1.8979
0.7500000	1.3585	0.2399	1.9246
1.3105349	1.7400	0.1750	2.4195

$\varphi = 1.353000$ rad.

μ	Re (a_2)	Im (a_2)	a_3
0.0000000	1.0943	0.0000	1.8179
0.2000000	1.1085	0.0925	1.8123
0.4000000	1.1520	0.1742	1.8006
0.6000000	1.2258	0.2340	1.7989
0.8000000	1.3299	0.2623	1.8335
1.0000000	1.4603	0.2551	1.9375
1.1000000	1.5330	0.2396	2.0254
1.2000000	1.6093	0.2182	2.1413
1.3680436	1.7432	0.1741	2.4034

$\varphi = 1.377000$ rad.

μ	Re (a_2)	Im (a_2)	a_3
0.0000000	1.0196	0.0000	1.7211
0.1500000	1.0278	0.0757	1.7166
0.3000000	1.0527	0.1467	1.7049
0.4500000	1.0949	0.2080	1.6912
0.6000000	1.1554	0.2546	1.6845
0.7500000	1.2340	0.2817	1.6975
0.9000000	1.3295	0.2867	1.7454
1.4321799	1.7497	0.1711	2.3918

$\varphi = 1.400000$ rad.

μ	Re (a_2)	Im (a_2)	a_3
0.0000000	0.9423	0.0000	1.6255
0.1900000	0.9557	0.1033	1.6162
0.3700000	0.9940	0.1920	1.5945
0.5600000	1.0635	0.2656	1.5714
0.7500000	1.1645	0.3073	1.5723
0.9300000	1.2870	0.3107	1.6255
1.0200000	1.3560	0.2994	1.6808
1.1200000	1.4369	0.2787	1.7695
1.2200000	1.5209	0.2517	1.8896
1.3100000	1.5981	0.2243	2.0257
1.4949916	1.7594	0.1658	2.3885

$\varphi = 1.422000$ rad.

μ	Re (a_2)	Im (a_2)	a_3
0.0000000	0.8625	0.0000	1.5321
0.1200000	0.8678	0.0713	1.5273
0.2400000	0.8842	0.1401	1.5138
0.3600000	0.9121	0.2037	1.4938

$\varphi = 1.422000$ rad.

μ	Re (a_2)	Im (a_2)	a_3
0.4800000	0.9525	0.2589	1.4712
0.6000000	1.0061	0.3026	1.4520
0.7200000	1.0726	0.3313	1.4441
0.8400000	1.1542	0.3424	1.4570
0.9600000	1.2462	0.3354	1.5011
1.0800000	1.3464	0.3123	1.5855
1.2000000	1.4515	0.2777	1.7166
1.5562835	1.7723	0.1581	2.3946

$\varphi = 1.444000$ rad.

μ	Re (a_2)	Im (a_2)	a_3
0.0000000	0.7759	0.0000	1.4378
0.1300000	0.7822	0.0839	1.4309
0.2600000	0.8016	0.1646	1.4112
0.3900000	0.8350	0.2384	1.3819
0.5200000	0.8842	0.3011	1.3487
0.6500000	0.9507	0.3475	1.3204
0.7800000	1.0351	0.3727	1.3093
0.9100000	1.1361	0.3733	1.3300
1.0400000	1.2494	0.3502	1.3970
1.1700000	1.3693	0.3098	1.5213
1.3000000	1.4913	0.2608	1.7073
1.6187370	1.7890	0.1475	2.4123

$\varphi = 1.466000$ rad.

μ	Re (a_2)	Im (a_2)	a_3
0.0000000	0.6815	0.0000	1.3438
0.1000000	0.6852	0.0705	1.3387
0.1900000	0.6949	0.1328	1.3259
0.2800000	0.7111	0.1929	1.3060
0.3700000	0.7343	0.2494	1.2801
0.4600000	0.7652	0.3008	1.2504
0.5600000	0.8098	0.3498	1.2160
0.6500000	0.8599	0.3841	1.1879
0.7500000	0.9274	0.4086	1.1659
0.8400000	0.9984	0.4162	1.1604
0.9400000	1.0872	0.4076	1.1782
1.0400000	1.1831	0.3827	1.2283
1.1200000	1.2624	0.3538	1.2954
1.1800000	1.3224	0.3289	1.3624
1.2400000	1.3822	0.3026	1.4438
1.3100000	1.4515	0.2715	1.5567
1.3700000	1.5103	0.2455	1.6682
1.4400000	1.5783	0.2167	1.8147
1.5000000	1.6362	0.1936	1.9536
1.5450000	1.6793	0.1773	2.0655
1.5900000	1.7223	0.1621	2.1836
1.6350000	1.7652	0.1478	2.3079
1.6823271	1.8102	0.1337	2.4449

μ	Re (a_2)	Im a_2	a_3
$\varphi = 1.483000$ rad.			
0.0000000	0.6022	0.0000	1.2724
0.1400000	0.6092	0.1053	1.2611
0.2800000	0.6311	0.2069	1.2283
0.4200000	0.6700	0.3002	1.1782
0.5600000	0.7300	0.3787	1.1187
0.7000000	0.8155	0.4332	1.0635
0.8400000	0.9290	0.4529	1.0337
0.9800000	1.0663	0.4314	1.0556
1.1200000	1.2149	0.3756	1.1527
1.2600000	1.3628	0.3052	1.3344
1.4000000	1.5054	0.2376	1.5963
1.7322273	1.8299	0.1207	2.4828
$\varphi = 1.500000$ rad.			
0.0000000	0.5162	0.0000	1.2034
0.1400000	0.5228	0.1127	1.1904
0.2800000	0.5436	0.2221	1.1524
0.4200000	0.5814	0.3239	1.0931
0.5600000	0.6410	0.4114	1.0202
0.7000000	0.7293	0.4739	0.9476
0.8400000	0.8517	0.4966	0.8989
0.9800000	1.0039	0.4681	0.9063
1.1200000	1.1675	0.3973	0.9992
1.2600000	1.3253	0.3127	1.1865
1.4000000	1.4732	0.2370	1.4588
1.7827784	1.8530	0.1054	2.5343
$\varphi = 1.518000$ rad.			
0.0000000	0.4160	0.0000	1.1347
0.0800000	0.4180	0.0693	1.1298
0.1600000	0.4238	0.1381	1.1152
0.2400000	0.4339	0.2059	1.0912
0.3200000	0.4488	0.2721	1.0582
0.4000000	0.4694	0.3357	1.0171
0.4800000	0.4968	0.3958	0.9692
0.5600000	0.5327	0.4504	0.9164
0.6400000	0.5792	0.4972	0.8614
0.7200000	0.6387	0.5325	0.8084
0.7800000	0.6931	0.5487	0.7734
0.8400000	0.7563	0.5537	0.7461
0.9100000	0.8404	0.5431	0.7288
0.9700000	0.9188	0.5193	0.7309
1.0400000	1.0134	0.4771	0.7574
1.1000000	1.0935	0.4328	0.8027
1.1600000	1.1704	0.3859	0.8687
1.2200000	1.2433	0.3400	0.9543
1.2800000	1.3126	0.2976	1.0576

μ	Re (a_2)	Im (a_2)	a_3
$\varphi = 1.518000$ rad.			
1.4700000	1.5155	0.1933	1.4811
1.6500000	1.6967	0.1298	1.9895
1.8369980	1.8815	0.0865	2.6073
$\varphi = 1.531000$ rad.			
0.0000000	0.3361	0.0000	1.0894
0.1000000	0.3387	0.0907	1.0810
0.2000000	0.3468	0.1808	1.0561
0.3000000	0.3612	0.2693	1.0151
0.4000000	0.3833	0.3550	0.9592
0.5000000	0.4157	0.4361	0.8902
0.6000000	0.4624	0.5090	0.8115
0.7000000	0.5295	0.5680	0.7290
0.8000000	0.6248	0.6022	0.6533
0.9000000	0.7535	0.5960	0.6022
1.0000000	0.9070	0.5388	0.5994
1.1000000	1.0606	0.4451	0.6630
1.2000000	1.1965	0.3483	0.7929
1.3000000	1.3156	0.2685	0.9769
1.4000000	1.4247	0.2081	1.2027
1.8765924	1.9051	0.0709	2.6744
$\varphi = 1.546000$ rad.			
0.0000000	0.2329	0.0000	1.0439
0.0400000	0.2332	0.0381	1.0425
0.0800000	0.2342	0.0763	1.0381
0.1200000	0.2358	0.1143	1.0307
0.1600000	0.2380	0.1523	1.0205
0.2000000	0.2410	0.1902	1.0074
0.2400000	0.2447	0.2279	0.9914
0.2800000	0.2493	0.2655	0.9725
0.3200000	0.2548	0.3029	0.9509
0.3600000	0.2614	0.3399	0.9265
0.4000000	0.2691	0.3766	0.8995
0.4400000	0.2782	0.4129	0.8698
0.5100000	0.2981	0.4748	0.8121
0.5800000	0.3247	0.5341	0.7476
0.6400000	0.3547	0.5814	0.6880
0.7000000	0.3940	0.6237	0.6259
0.7700000	0.4560	0.6627	0.5533
0.8300000	0.5282	0.6813	0.4954
0.8900000	0.6223	0.6776	0.4480
0.9600000	0.7581	0.6333	0.4179
1.1600000	1.1275	0.3587	0.5693
1.3500000	1.3562	0.1935	0.9602
1.5400000	1.5532	0.1164	1.4743
1.7300000	1.7440	0.0753	2.0778
1.9227228	1.9361	0.0502	2.7710

μ	Re (a_2)	Im (a_2)	a_3
	$\varphi = 1.553000$ rad.		
0.0000000	0.1790	0.0000	1.0263
0.3000000	0.1942	0.2904	0.9413
0.4000000	0.2080	0.3857	0.8758
0.5000000	0.2291	0.4791	0.7929
0.6000000	0.2617	0.5688	0.6938
0.7000000	0.3139	0.6508	0.5818
0.8000000	0.4034	0.7137	0.4645
0.9000000	0.5636	0.7241	0.3619
1.0000000	0.8094	0.6140	0.3234
1.9444109	1.9522	0.0392	2.8249
	$\varphi = 1.560000$ rad.		
0.0000000	0.1194	0.0000	1.0119
0.0200000	0.1194	0.0197	1.0115
0.0400000	0.1195	0.0395	1.0103
0.0600000	0.1198	0.0592	1.0084
0.0800000	0.1201	0.0790	1.0057
0.1000000	0.1205	0.0987	1.0021
0.1200000	0.1209	0.1184	0.9978
0.1400000	0.1215	0.1382	0.9928
0.1600000	0.1222	0.1579	0.9869
0.1800000	0.1229	0.1776	0.9803
0.2000000	0.1238	0.1973	0.9729
0.2200000	0.1248	0.2170	0.9647
0.2700000	0.1277	0.2661	0.9408
0.3300000	0.1323	0.3250	0.9058
0.3900000	0.1384	0.3837	0.8640
0.4500000	0.1462	0.4420	0.8153
0.5200000	0.1583	0.5095	0.7501
0.5900000	0.1750	0.5760	0.6761
0.7900000	0.2801	0.7483	0.4241
0.9800000	0.6998	0.6964	0.2190
1.1800000	1.1634	0.2304	0.4587
1.3800000	1.3812	0.1047	0.9415
1.5700000	1.5733	0.0619	1.4925
1.7700000	1.7736	0.0396	2.1555
1.9661996	1.9696	0.0268	2.8858
	$\varphi = 1.564000$ rad.		
0.0000000	0.0814	0.0000	1.0056
0.1650000	0.0835	0.1640	0.9787
0.2200000	0.0852	0.2186	0.9578
0.2750000	0.0875	0.2731	0.9309
0.3300000	0.0905	0.3276	0.8981
0.3850000	0.0943	0.3821	0.8593
0.4400000	0.0991	0.4364	0.8147
0.4950000	0.1053	0.4905	0.7642
0.5500000	0.1133	0.5443	0.7079

μ	Re (a_2)	Im (a_2)	a_3
	$\varphi = 1.564000$ rad.		
0.6050000	0.1236	0.5978	0.6459
1.9786950	1.9803	0.0187	2.9246
	$\varphi = 1.567000$ rad.		
0.0000000	0.0499	0.0000	1.0021
0.1290000	0.0507	0.1287	0.9856
0.1720000	0.0513	0.1716	0.9727
0.2150000	0.0521	0.2145	0.9561
0.2580000	0.0532	0.2573	0.9359
0.3010000	0.0545	0.3002	0.9120
0.3440000	0.0561	0.3430	0.8844
0.3870000	0.0580	0.3859	0.8531
0.4300000	0.0603	0.4287	0.8182
0.4730000	0.0631	0.4714	0.7796
1.9880879	1.9887	0.0118	2.9560
	$\varphi = 1.570000$ rad.		
0.0000000	0.0130	0.0000	1.0001
0.0020000	0.0130	0.0020	1.0001
0.0040000	0.0130	0.0040	1.0001
0.0060000	0.0130	0.0060	1.0001
0.0080000	0.0130	0.0080	1.0001
0.0100000	0.0130	0.0100	1.0000
0.0120000	0.0130	0.0120	1.0000
0.0140000	0.0130	0.0140	1.0000
0.0160000	0.0130	0.0160	0.9999
0.0180000	0.0130	0.0180	0.9998
0.0200000	0.0130	0.0200	0.9997
0.0220000	0.0130	0.0220	0.9997
0.0260000	0.0130	0.0260	0.9995
0.0300000	0.0130	0.0300	0.9992
0.0340000	0.0130	0.0340	0.9990
0.0380000	0.0130	0.0380	0.9987
0.0490000	0.0130	0.0490	0.9977
0.0600000	0.0130	0.0600	0.9965
0.0800000	0.0130	0.0800	0.9938
0.1000000	0.0131	0.1000	0.9902
0.1500000	0.0132	0.1500	0.9777
0.2000000	0.0135	0.2000	0.9602
0.3000000	0.0142	0.2999	0.9102
0.4000000	0.0153	0.3999	0.8402
0.6000000	0.0198	0.5998	0.6403
0.8000000	0.0345	0.7994	0.3607
1.0000000	0.7141	0.7001	0.0243
1.2000000	1.1998	0.0268	0.4408
1.4000000	1.4000	0.0119	0.9605
1.6000000	1.6000	0.0070	1.5603
1.8000000	1.8000	0.0046	2.2403
1.9974989	1.9975	0.0033	2.9902

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Colloquium Publications

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1. H. S. White, *Linear Systems of Curves on Algebraic Surfaces;*
   F. S. Woods, *Forms of Non-Euclidean Space;*
   E. B. Van Vleck, *Selected Topics in the Theory of Divergent Series and of Continued Fractions;*
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