A FORMALIZATION OF SET THEORY WITHOUT VARIABLES

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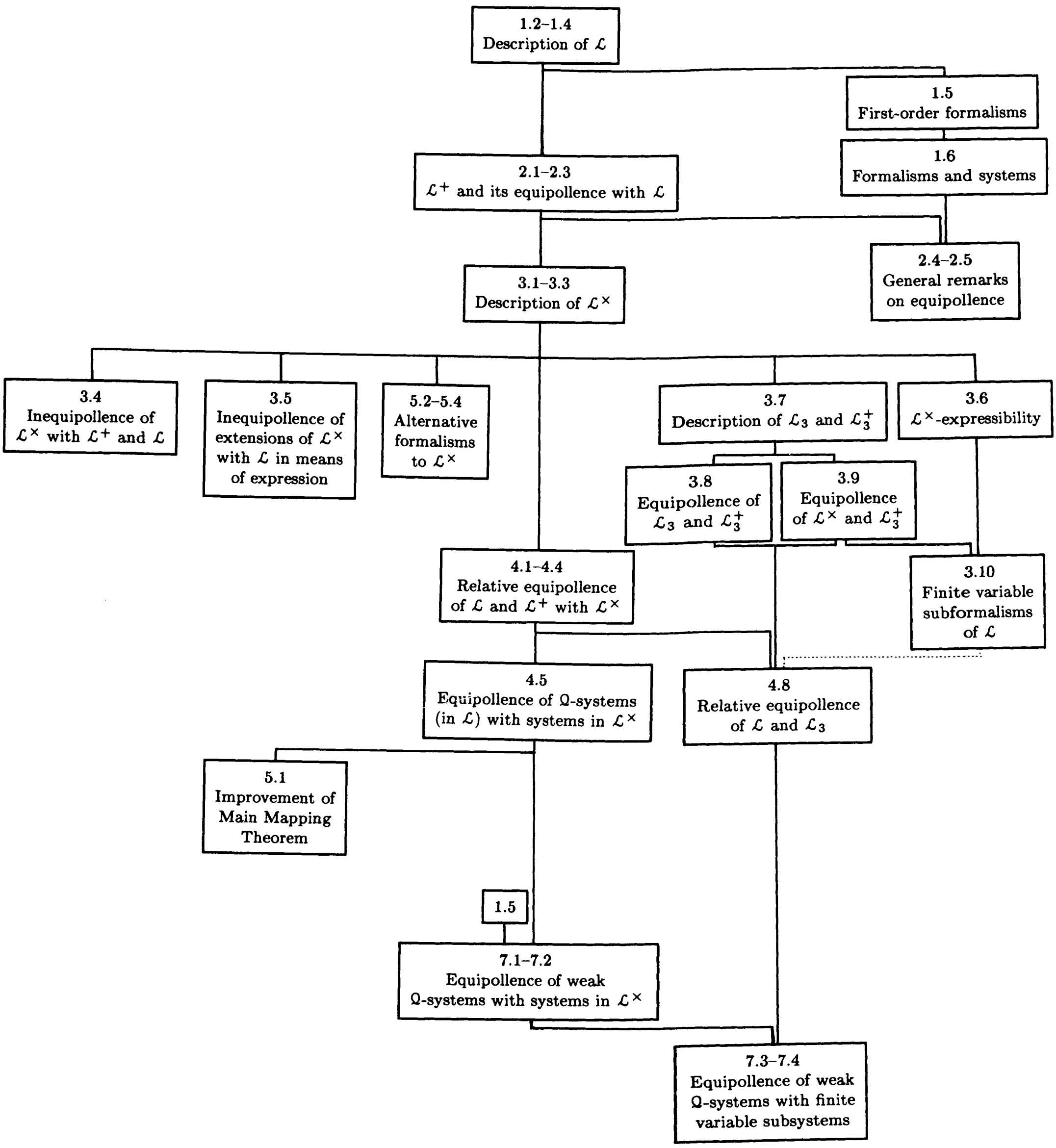
Explanation of section interdependence diagrams

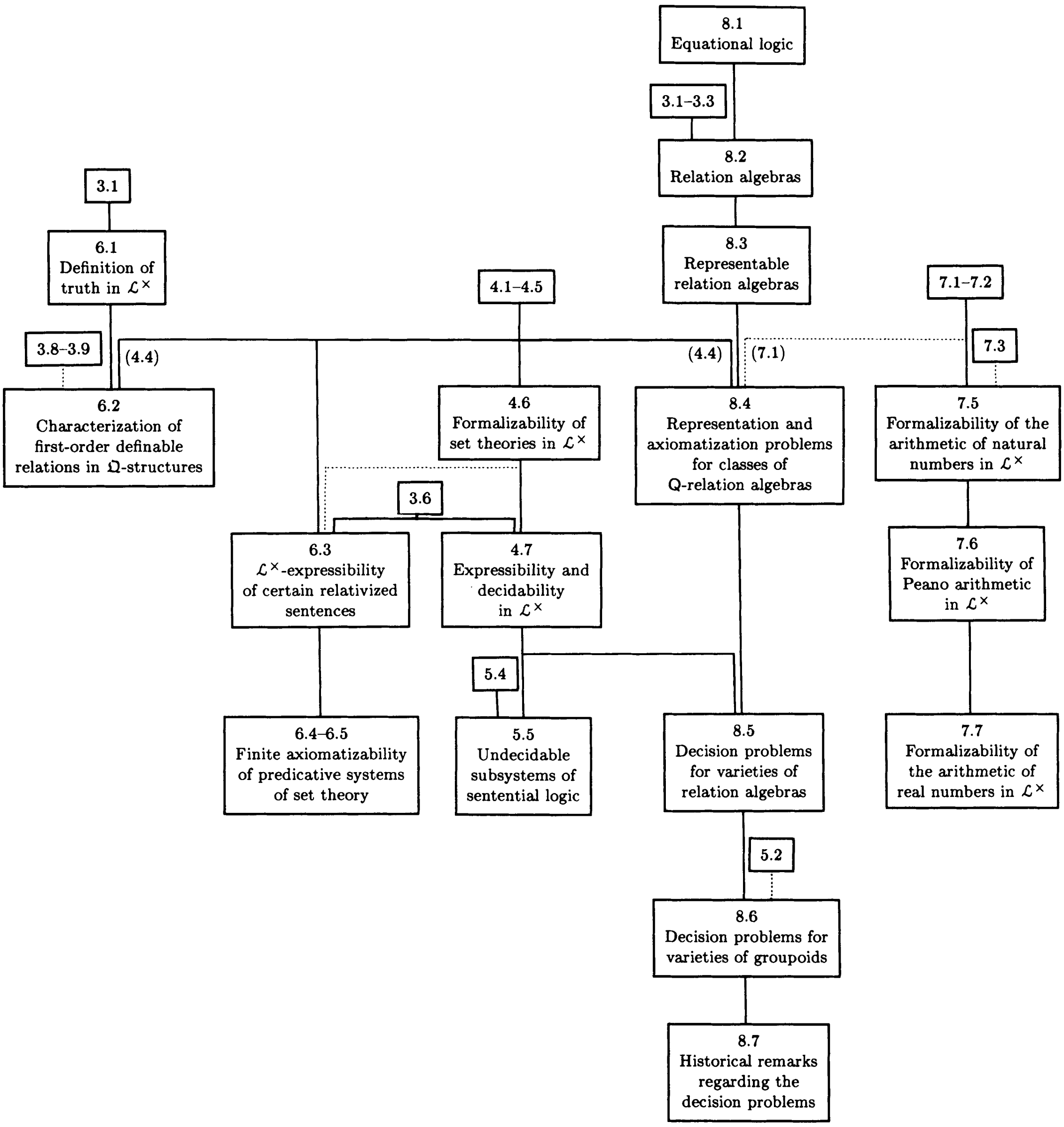
The diagrams on the next two pages indicate the essential interdependencies of the various sections of this book. In general, the dependence of a section on earlier sections is determined by following upwards the lines leading to the section's box. For example, Section 4.8 depends on Sections 3.8–3.9 (and possibly on sections above them, such as 3.7, 3.1–3.3, 2.1–2.3, and 1.2–1.4), as well as on Sections 4.1–4.4 (and possibly on sections above them). A small part of it also depends on part of Section 3.10; this more limited dependence is indicated by a dotted line. For a second example, Section 4.7 depends on 3.6 (which in turn depends on earlier sections), as well as on 4.6 (and possibly on some of the sections above it, such as 4.1–4.5). As a final example, Section 6.2 depends on 6.1 and 4.4. (A line flows to 6.2 from the box labeled 4.1–4.5, but we have indicated parenthetically that only 4.4 is really important.) A small part of 6.2 also uses some results from 3.8–3.9.

Diagrams of Section Interdependence

Equipollence Results



Applications



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Preface

In this work we shall show that set theory and number theory can be developed within the framework of a new, different, and very simple formalism, $\mathcal{L}^\times$. $\mathcal{L}^\times$ is closely related to the equational theory of abstract relation algebras essentially given in Chin–Tarski [**1951**]. Its language contains no variables, quantifiers, or sentential connectives. There are two basic symbols, $\mathring{\mathbf{1}}$ and $\boldsymbol{E}$, intended to denote the identity and the (set-theoretic) membership relations. Compound expressions are constructed from the basic symbols by means of four operation symbols, $\odot$, $\smile$, $+$, and $^-$, that denote the well-known operations (on and to binary relations) of relative product, conversion, Boolean addition, and complementation. All mathematical statements in $\mathcal{L}^\times$ are formulated as (variable-free) equations between such expressions. The deductive apparatus of $\mathcal{L}^\times$ is based upon ten logical axiom schemata that are the analogues of the equational postulates for abstract relation algebras essentially given in Chin–Tarski [**1951**], p. 344. There is just one rule of inference, namely, the rule familiar from high school algebra of replacing equals by equals. A (deductive) system in $\mathcal{L}^\times$ is given by a set of nonlogical axioms, i.e., equations of $\mathcal{L}^\times$, and can be identified with the theory in $\mathcal{L}^\times$ generated by these axioms, i.e., with the set of all equations derivable from these axioms, the logical axioms of $\mathcal{L}^\times$, and equations of the form $A = A$, by means of the single rule of replacing equals by equals.

$\mathcal{L}^\times$ appears to be quite weak in its powers of expression and proof. Even the simple statement that there exist at least four elements cannot be equivalently expressed in $\mathcal{L}^\times$, as follows at once from a result of Korselt given in Löwenheim [**1915**], p. 448 (see below). Furthermore, $\mathcal{L}^\times$ is *semantically incomplete* in the sense that there are semantically valid equations in $\mathcal{L}^\times$ which are not derivable. This follows readily from a result of McKenzie [**1970**] (sharpening an earlier result of Lyndon [**1950**]) by which the postulates for abstract relation algebras given in Chin–Tarski [**1951**] do not even yield all of the equations with just one variable that are true in every concrete algebra of relations. We shall show, in fact, that $\mathcal{L}^\times$ is equipollent (in a natural sense) to a certain fragment, $\mathcal{L}_3$, of first-order logic having one binary predicate and containing *just three variables.* ($\mathcal{L}_3$ is also semantically incomplete.) It is therefore quite surprising that $\mathcal{L}^\times$ proves adequate for the formalization of practically all known systems of set theory, and hence for the development of all of classical mathematics.

As a language suitable for the formalization of most set-theoretical systems, we take the first-order logic $\mathcal{L}$ with equality and one nonlogical binary predicate $\boldsymbol{E}$. (For technical reasons we use "$\mathring{\mathbf{1}}$" instead of "$=$" as the name of the symbol denoting the relation of equality between individuals.) A system in $\mathcal{L}$ is given by a set of nonlogical axioms, and, as before, can be identified with the theory in $\mathcal{L}$ generated by these axioms.

It proves convenient to consider also an auxiliary formalism $\mathcal{L}^+$ that is a kind of definitional extension of $\mathcal{L}$. In addition to the basic predicates $\mathring{\mathbf{1}}$ and $\boldsymbol{E}$ of $\mathcal{L}$, the vocabulary of $\mathcal{L}^+$ contains as logical constants (of a new kind) the symbols $\odot$, $^\smile$, $+$, and $^-$ from $\mathcal{L}^\times$, by means of which (compound) *predicates* are constructed from $\mathring{\mathbf{1}}$ and $\boldsymbol{E}$; specifically, if A and B are predicates, then so are $A \odot B$, $A^\smile$, $A + B$, and A^-. The vocabulary of $\mathcal{L}^+$ also contains the second equality symbol, $=$, from $\mathcal{L}^\times$, intended to denote the relation of equality between binary relations. Atomic formulas of $\mathcal{L}^+$ are expressions of the form xAy and $A = B$, where x, y are individual variables and A, B are predicates. Arbitrary formulas are constructed from atomic ones in the usual way. In addition to a set of logical axioms similar in character to those of $\mathcal{L}$, $\mathcal{L}^+$ has five axiom schemata that can be regarded as possible definitions of the constants $\odot$, $^\smile$, $+$, $^-$, and $=$. For example, the schemata for $\odot$ and $=$ are respectively

$$\forall_{xy}(xA \odot By \leftrightarrow \exists_z(xAz \wedge zBy)),$$

and

$$A = B \leftrightarrow \forall_{xy}(xAy \leftrightarrow xBy),$$

where A, B are arbitrary predicates of $\mathcal{L}^+$. The rules of inference for $\mathcal{L}^+$ and the notion of a system in $\mathcal{L}^+$ are taken just as in $\mathcal{L}$. A "definitional extension" of $\mathcal{L}$ which essentially includes $\mathcal{L}^+$ is discussed in Quine [**1969**], pp. 15–27, under the name of "the virtual theory of classes".

With the help of $\mathcal{L}^+$ we shall compare the powers of expression and proof of $\mathcal{L}$ and $\mathcal{L}^\times$, and also of systems developed in these formalisms. Since $\mathcal{L}^\times$ has no variables, we must replace familiar notions like "definitionally equivalent" by suitable analogues. In each of the formalisms $\mathcal{L}$, $\mathcal{L}^+$, and $\mathcal{L}^\times$ (and, more generally, in every system developed in these formalisms) there is the notion of *sentence* and the notion of *derivability*, i.e., of a *sentence being derivable* (in the formalism or system) *from a set of sentences.* Suppose $\mathcal{S}_1$ and $\mathcal{S}_2$ are formalisms (or systems) that have these two notions. We say that $\mathcal{S}_2$ is an *extension* of $\mathcal{S}_1$ if every sentence of $\mathcal{S}_1$ is a sentence of $\mathcal{S}_2$ and derivability in $\mathcal{S}_1$ implies derivability in $\mathcal{S}_2$. Such an extension $\mathcal{S}_2$ is called an *equipollent extension* of $\mathcal{S}_1$ if also the following two conditions hold: (1) (equipollence in means of expression) for every sentence X of $\mathcal{S}_2$ there is a sentence Y of $\mathcal{S}_1$ that is equivalent to X in $\mathcal{S}_2$, i.e., Y is derivable from X, and X from Y, in $\mathcal{S}_2$; (2) (equipollence in means of proof) for every sentence X and set of sentences Ψ of $\mathcal{S}_1$, if X

is derivable from Ψ in $\mathcal{S}_2$, then it is so derivable in $\mathcal{S}_1$. (We avoid the term "definitional extension" because it usually involves a version of (1) applying to arbitrary formulas, and not just to sentences.) Finally, $\mathcal{S}_1$ and $\mathcal{S}_2$ are said to be *equipollent* if they have a common equipollent extension. When we wish to emphasize the role of a particular common equipollent extension $\mathcal{S}_3$, we shall say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are *equipollent relative to* $\mathcal{S}_3$. It is not difficult to show that when $\mathcal{S}_1$ and $\mathcal{S}_2$ are equipollent, there is a natural one-one correspondence between the theories (i.e., the deductively closed sets of sentences) in $\mathcal{S}_1$ and $\mathcal{S}_2$ that preserves various important properties of theories such as consistency and completeness (cf. Theorem 2.4(viii) below); a theory is *consistent* if it does not coincide with the set of all sentences, and it is *complete* if it is a maximal consistent set of sentences.

It is readily seen (and follows from what is in Quine [**1969**]) that $\mathcal{L}^+$ is an equipollent extension of $\mathcal{L}$ (and even more; cf. §2.3). It is also easy to show (using, e.g., the semantic completeness of $\mathcal{L}^+$) that $\mathcal{L}^+$ is an extension of $\mathcal{L}^\times$. However, it is not an equipollent extension, and in fact both (1) and (2) fail. The failure of (1) is a direct consequence of Korselt's result, cited above (see Theorem 3.4(iv)), while the failure of (2) is due to the aforementioned semantic incompleteness of $\mathcal{L}^\times$ (see Theorem 3.4(vi)). Regarding (1), we shall actually prove (in Theorem 3.5(viii)) the following much stronger result.

(i) *Even if we enrich $\mathcal{L}^\times$ (and $\mathcal{L}^+$)by adjoining any finite number of new constants, all of which are intended to denote operations on and to binary relations, or else relations between binary relations (over the universe of any realization of $\mathcal{L}^\times$), and which are "logical" in the sense that the denoted operations and relations are preserved under all permutations of the universe, there will still be sentences of $\mathcal{L}$ that are not equivalent (in the enriched $\mathcal{L}^+$)with any sentence of the enriched $\mathcal{L}^\times$.*

Thus, the inadequacy of the expressive powers of $\mathcal{L}^\times$ is not due to a faulty choice of the set of fundamental notions; there is no way of extending this set in a finite and "logical" way so as to achieve equipollence with $\mathcal{L}$ in means of expression. We shall also prove (in §§3.8 and 3.9) the theorem, referred to before, that:

(ii) *$\mathcal{L}^\times$ is equipollent to a certain three-variable fragment, $\mathcal{L}_3$, of $\mathcal{L}$ (relative to a similar fragment, $\mathcal{L}_3^+$, of $\mathcal{L}^+$).*

As we have seen, $\mathcal{L}^\times$ is weaker than $\mathcal{L}$ both in means of expression and proof. Nevertheless, as stated above, we are going to establish the surprising result that $\mathcal{L}^\times$ is adequate for the development of classical mathematics. This will follow from two theorems, (iii) and (iv), which we now describe.

In (iii) we shall show that for certain special systems in $\mathcal{L}$ called $\mathcal{Q}$-systems we can construct equipollent systems in $\mathcal{L}^\times$. A system $\mathcal{S}$ in $\mathcal{L}$ is called a $\mathcal{Q}$-system if there are formulas D and E (of $\mathcal{L}$) containing at most three distinct variables, and just two free variables, such that in every model of $\mathcal{S}$, the two binary relations defined by D and E form a pair of conjugated quasiprojections, i.e., are functions with the following additional property: for every pair of elements x, y (in the universe of the model) there is a z which is mapped to x by the first function and to y by the second; the element z should be thought of as representing the ordered pair $\langle x, y\rangle$. Our main equipollence theorem (which is established in §§4.4 and 4.5, and upon which most of the later results in the book are based) is as follows.

(iii) *Every $\mathcal{Q}$-system $\mathcal{S}$ in $\mathcal{L}$ is equipollent with a system in $\mathcal{L}^{\times}$ (relative to a system in $\mathcal{L}^{+}$); moreover, the system in $\mathcal{L}^{\times}$ will be, e.g., finitely axiomatizable or decidable iff $\mathcal{S}$ is.*

In (iv) this theorem is generalized to *weak* $\mathcal{Q}$-systems developed in *arbitrary* first-order formalisms with finitely many nonlogical constants. The definition of a weak $\mathcal{Q}$-system is obtained from that of a $\mathcal{Q}$-system by dropping the restriction on the number of distinct variables occurring in formulas D and E. We prove, in fact (in Theorem 7.2(iv)), that:

(iv) *Every weak $\mathcal{Q}$-system $\mathcal{U}$ developed in a first-order formalism with finitely many nonlogical constants is equipollent with a system in $\mathcal{L}^{\times}$; again, this latter system is, e.g., finitely axiomatizable or decidable iff $\mathcal{U}$ is.*

Both (iii) and (iv) seem very specialized. However, we shall show (in §4.6) that the hypothesis of (iii) holds for almost every known system of set theory, and (in §§7.5–7.7) that the hypothesis of (iv) applies, e.g., to the (full) elementary theory of natural numbers, to its well-known, recursively axiomatized subtheory, first-order Peano arithmetic, and to the elementary theory of the real numbers (with the set of natural numbers as a distinguished subset). Thus each of these systems is equipollent to a system in $\mathcal{L}^{\times}$.

With the help of the equipollence theorems (ii)–(iv) we shall also investigate a variety of other problems, quite apart from the one of formalizing mathematical systems in $\mathcal{L}^{\times}$. These concern, for example, the construction of undecidable subsystems of sentential logic (in §5.5), the relatively simple definition of truth for the formalism $\mathcal{L}^{\times}$ (in 6.1(i),(ii)), a characterization of the first-order definable binary relations in models of set theory and arithmetic (in 6.2(ix) and 7.4), the finite axiomatizability of predicative versions of systems of set theory (in 6.4(vi) and 6.5(iv)), the adequacy of first-order formalisms with only finitely many variables for the development of various mathematical disciplines (in 4.8(xi),(xii), 7.3(ii),(iii), and 7.5(vi)), the first-order definitional equivalence of number theory and the theory of hereditarily finite sets (in 7.5(v) and 7.6(ii)), the representation problem for relation algebras with a pair of quasiprojective elements, and the nonfinite axiomatizability of the equational theory of these algebras (in 8.4(iii),(vii)), the undecidability of the equational theory of several important classes of relation algebras (in 8.5(xii)), and what seems to be the first construction of a finitely based, essentially undecidable equational theory (and in fact a theory of groupoids—see 8.5(xi) and 8.6(x)).

We now make some historical remarks regarding the above theorems and their relation to results in the literature. The mathematics of the present work is rooted in the calculus of relations (or the calculus of relatives, as it is sometimes called) that originated in the work of A. De Morgan, C. S. Peirce, and E. Schröder during the second half of the nineteenth century. The universe of discourse of this calculus is the collection of all binary relations on an arbitrary but fixed set U, i.e., the set of all subsets of $U \times U$. There are six fundamental operations on and to (binary) relations, and four distinguished relations. Specifically, there are the four binary operations of forming, for any two relations R and S, their absolute sum, which is simply the union $R \cup S$, their absolute product, which is the intersection $R \cap S$, their relative sum, $R \dagger S$, consisting of all pairs $\langle x, y \rangle$ such that for every z either xRz or zSy (i.e., either $\langle x, z \rangle$ is in R or $\langle z, y \rangle$ is in S), and their relative product, $R|S$, consisting of all pairs $\langle x, y \rangle$ such that for some z, both xRz and zSy. Further, there are two unary operations of forming, for every relation R, its complement, $\sim R$, with respect to $U \times U$, and its converse, R^{-1}, consisting of all pairs $\langle x, y \rangle$ such that yRx. Finally, the distinguished relations are the absolute zero, which is the empty relation $\varnothing$, the absolute unit, which is the universal relation $U \times U$, the relative zero, which is the diversity relation Di on U consisting of all pairs $\langle x, y \rangle$ such that $x \neq y$, and the relative unit, which is the identity relation Id on U consisting of all pairs $\langle x, y \rangle$ such that $x = y$. This is the framework of the calculus as finally presented in Peirce [**1882**] after several earlier versions.

Both Peirce and, later, Schröder, who extended Peirce's work in a very thorough and systematic way in Schröder [**1895**], were interested in the expressive powers of the calculus of relations and the great diversity of laws that could be proved. They were aware that many elementary statements about (binary) relations can be expressed as equations in this calculus. (By an "elementary statement" about relations $R, S, \ldots$ (over U) we mean a first-order statement about the structure $\langle U,\ R, S, \ldots \rangle$.) To give an example, the (elementary) statement that a relation R is transitive,

$$\textit{for every}\quad x, y, z, \quad \textit{if}\quad xRy \quad \textit{and}\quad yRz, \quad \textit{then}\quad xRz,$$

can be expressed by the equation

$$(R|R) \cup R = R.$$

Similarly, the more complicated statement that R is a one-one function mapping U onto itself is rendered by the equation

$$[(R|R^{-1}) \cap Di] \cup [(R^{-1}|R) \cap Di] \cup [\sim R \dagger \varnothing] \cup [\varnothing \dagger \sim R] = \varnothing.$$

Schröder seems to have been the first to consider the question whether all elementary statements about relations are expressible as equations in the calculus of relations, and in Schröder [**1895**], p. 551, he proposed a positive solution to the problem. A critique of Schröder's proposed solution appeared in Löwenheim [**1915**], p. 450, along with a negative solution due to Korselt that was referred to above. A far-reaching extension of Korselt's result, formulated in (i) above for $\mathcal{L}^{\times}$ (but also true in the more general setting of the calculus of relations), was first announced in Tarski [**1941**].

In the same paper Tarski posed the problem of proving that there is no algorithm for deciding in every particular case whether an elementary statement about relations is expressible in the calculus (as an equation). Michael Kwatinetz finally settled the problem around 1971, with the help of (ii) above (restated for the calculus of relations), by showing that the set of elementary statements which can equivalently be formulated using just three variables is not recursive (see Kwatinetz [**1981**] for the proof).

Despite the weak expressive powers of the calculus of relations, Tarski was able to establish a kind of relative equipollence in means of expression between it and the elementary theory of relations. Namely, if we assume we have a pair of conjugated quasiprojections, then for any elementary statement X we can effectively construct an equation X^* in the calculus that is equivalent to X. This is a preliminary and much weaker form of (iii) above; it does not concern itself with the problem of equipollence in means of proof. With its help, Tarski proved that any decision procedure for the set of true equations in the calculus of relations would bring with it a decision procedure for the elementary theory of relations, in contradiction to a result of Church [**1936**] and Kalmár [**1936**]; hence the set of true equations in the calculus of relations is not recursive (see 8.5(xii) below). This theorem was announced in Tarski [**1941**], p. 88. Lemmas I–III of the abstract Tarski [**1953**] give a rough outline of Tarski's original proof.

Tarski [**1941**] presented an interesting formalization of the calculus of relations as a deductive discipline. The language contained (binary) relation variables, but no individual variables or quantifiers, and although sentential connectives were present, it was pointed out on p. 87 of *op. cit.* that an equivalent formalization involving only equations, i.e., without sentential connectives, could be given. (Such a formalization was essentially carried out in Chin–Tarski [**1951**].) Tarski proposed a finite set of axioms for the calculus (essentially equivalent to the set of axioms given in Chin–Tarski [**1951**], as noted in *ibid.*, p. 352, footnote 10), indicated that he could derive all the hundreds of laws occurring in Schröder [**1895**] on the basis of these axioms, and asked whether every true law (true for all domains of individuals) in the calculus was so derivable. As mentioned above, this problem was subsequently answered negatively by Lyndon [**1950**], and, in fact, Monk [**1964**] showed that the set of true equations of the calculus is not finitely axiomatizable at all.

Nevertheless, just as in the case of expressibility, Tarski was able to establish a kind of relative equipollence in means of proof between his axiomatization of the calculus and the elementary theory of relations. This is essentially the result stated above in (iii) (when reformulated for the calculus of relations). From this Tarski concluded that the set of equations derivable from his set of axioms for the calculus of relations is not recursive (see 8.5(xii)). Further, since his theorem reduced every problem concerning the derivability of a mathematical statement from a set of axioms to the problem of whether an equation is derivable from a set of equations in the calculus of relations, in principle the whole of mathematical research could be carried out within the framework of this calculus. These theorems were obtained by Tarski during the period 1943–1944, and presented for the first time in his seminar on relation algebras at the University of California, Berkeley, during the year 1945. References to these theorems, as well as to the Berkeley seminar, can be found in Chin–Tarski [**1951**], pp. 341–343; see also Chin [**1948**], pp. 2–3. The abstracts Tarski [**1953**], [**1953a**], [**1953b**], [**1954**], [**1954a**] contain announcements of these theorems and several of the other results (also dating from the 1943–1944 period) that were referred to in the first part of this foreword.

Roughly speaking, the formalism $\mathcal{L}^{\times}$ that is the central focus of this work is obtained from Tarski's equational formalization of the calculus of relations by introducing the constant $\boldsymbol{E}$ and deleting all variables.

Tarski's formalization of set theory in $\mathcal{L}^{\times}$ was certainly not the first attempt to eliminate the use of variables in formalizing mathematics. Probably the earliest results in this direction appeared in Schönfinkel [**1924**]. There, a kind of calculus of unary functions was developed. Basically, Schönfinkel considered three distinguished unary functions (later called combinators by Curry), C, S, and U, and one binary operation on and to unary functions: that of applying a unary function f to an argument x, the result being represented by juxtaposing the two, as in "fx". The definitions of C, S, and U are not simple. Each of them has the property that, when applied to a unary function, it yields another unary function; thus each of them takes on unary functions as both arguments and values. For example, C is the function that, when applied to any unary function f, yields a constant unary function, and in fact the function constantly equal to f, i.e., Cf is the unary function which, when applied to any argument x, yields f, in symbols $(Cf)x = f$. The definitions of S and U are still more involved; the reader is referred to Schönfinkel's paper. By means of the binary operation of functional application we can construct further unary functions (called compound combinators) from the basic three, for example, CC, CS, $(CS)C$, etc. Schönfinkel indicated how not only bound individual variables, but also bound variables of higher orders can be eliminated from mathematical statements with the help of combinators. Thus the expressive power of his calculus reaches far beyond the domain of first-order logic.

Schönfinkel made no attempt to set up a deductive apparatus for his calculus. This task was taken up by Curry and his collaborators, starting in the late 1920s, and proved to be quite involved. We shall not attempt to describe their many achievements—the reader is referred to books on combinatory logic (as this domain is now called), and in particular to Curry–Feys [**1958**] and Curry–Hindley–Seldin [**1972**], which contain extensive bibliographies. Rather, we shall briefly contrast the character of the results presented in this book with those that have been obtained in the domain of combinatory logic; moreover, we contrast them only as regards the specific problem of developing parts of mathematics within a variable-free formalism.

First of all, in contrast to the expressive powers of combinatory logic, those of $\mathcal{L}^{\times}$ do not overreach first-order logic, and (as was pointed out above) actually comprise but a weak part of it, namely the first-order logic of three variables. Only under certain additional assumptions (satisfied by most systems of set theory and various systems of arithmetic) does $\mathcal{L}^{\times}$ become equipollent with first-order logic in means of expression. Similar remarks apply to the deductive powers of $\mathcal{L}^{\times}$. Secondly, the method presented in this book for formalizing a given first-order system within $\mathcal{L}^{\times}$ is quite general; it can be applied almost mechanically to many different mathematical theories. In contrast to this, the various attempts to develop different parts of mathematics within combinatory logic have been quite specific in character, and the approaches used have depended on the particular theory to be formalized. Finally, each of our correlated systems in $\mathcal{L}^{\times}$ is shown to be equipollent with the original first-order system in a strong and precisely defined sense that entails, e.g., the equiconsistency, equicompleteness, and equidecidability of the two systems. It is not at all clear to what extent various first-order systems and their combinatory analogues are equipollent. Indeed the very problem of the consistency, or relative consistency, of systems formalized in combinatory logic has traditionally posed difficulties; in several cases the answer proved to be negative and some of these problems are still open.

In the late 1940s and early 1950s there began some work which has a bearing on the problem of formalizing mathematics without variables. Various algebraic theories were developed that are analogues of first-order logic, much as Boolean algebra is an algebraic analogue of the sentential calculus. The creation of the theories of relation algebras by Tarski, and of projective algebras by Everett-Ulam [**1946**] may be viewed as preliminary steps in this direction. The theory of cylindric algebras, perhaps the most extensively developed of such theories, was created by Tarski in collaboration with his former students Louise Chin (Lim) and Frederick Thompson during the period 1948–52, and further developed by Tarski, Henkin, and Monk. A detailed presentation of various portions of this theory can be found in Henkin–Monk–Tarski [**1971**], [**1985**]. The closely related theory of polyadic algebras was created by Halmos in the mid 1950s; the relevant papers can be found in Halmos [**1962**]. Other noteworthy theories of this type can be found in Bernays [**1959**] and Craig [**1974**]. The specific problem of using such algebraic theories to construct formalisms which contain no (bound) variables, quantifiers, or sentential connectives, and which are equipollent in some sense to first-order logic, is addressed in these last two works and in Quine [**1960**], [**1971**]. An excellent discussion of various approaches to this problem can be found in Quine [**1971**].

With the exception of some algebraic notions and results in Chapter 8, this work is intended to be largely self-contained, and accessible not only to mathematicians and logicians, but also to computer scientists, philosophers, and others who may be interested in foundational research.

In §§1.2–1.5, 2.1–2.3, and 3.1–3.4 we respectively describe the formalisms $\mathcal{L}$, $\mathcal{L}^+$, and $\mathcal{L}^\times$, and their interrelationship. After reading these portions of the book, it is possible to proceed directly to the main equipollence results presented in §§4.1–4.5 and 7.1–7.2, omitting the intervening text. §§4.6 and 7.5–7.7, concerning the formalizability of various systems of set theory and arithmetic in $\mathcal{L}^\times$, essentially depend only on §§4.1–4.5 and 7.1–7.2, respectively. A more detailed picture of the interdependence of different sections of the book is presented in the diagrams following the table of contents.

Alfred Tarski
Steven Givant
Berkeley, California
October, 1983

Postscript

Alfred Tarski died on October 27, 1983, shortly after the manuscript for this work was completed. With his passing, the world has lost a great logician and an inspiring teacher, and I have lost a loyal friend. In the period since his death it has become apparent that certain small additions should be made to the text. For example, in the last few years a number of interesting results have been obtained that have a direct bearing on some of the open problems stated here. In particular, after receiving preliminary copies of the manuscript, Roger Maddux, Hajnal Andréka, and István Németi solved several of these open problems. In addition, various relevant results in the literature that appeared in the late 1970s and early 1980s, and were overlooked by us, have been called to my attention by Andréka, Maddux, and Németi as well as by George McNulty. Rather than amending the text itself at this point, I have decided to include some additional footnotes, indicated by an asterisk (*), to discuss these results.

Steven Givant
Berkeley, California
January, 1986

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Acknowledgments

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