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Fully Nonlinear Elliptic Equations

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Fully Nonlinear Elliptic Equations

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ABSTRACT. This book provides a self-contained development of the regularity theory of solutions of fully nonlinear elliptic equations. These partial differential equations arise in control theory and optimization. The goal of this work is to extend the classical Schauder and Calderón-Zygmund regularity theories for linear elliptic equations to the fully nonlinear context. The book contains a detailed presentation of all the techniques needed. We do not treat them in their greatest generality; rather we present the key ideas and prove all the results needed for the subsequent theory.

We develop the theory of viscosity solutions of nonlinear equations, the Alexandroff estimate and Krylov-Safonov Harnack inequality for viscosity solutions, Jensen's uniqueness theory for viscosity solutions, Evans and Krylov regularity theory for convex fully nonlinear equations, and finally the regularity theory for fully nonlinear equations with variable coefficients.

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Bibliography

[B] Brezis, H., *Analyse Fonctionnelle: Théorie et Applications*, Paris, New York, Masson, 1983.

[Ca] Cabré, X., *On the Alexandroff-Bakelman-Pucci estimate and the reversed Hölder inequality for solutions of elliptic and parabolic equations*, to appear in Comm. Pure Appl. Math.

[C1] Caffarelli, L. A., *Interior a priori estimates for solutions of fully nonlinear equations*, Annals of Mathematics 130 (1989), 189-213.

[C2] Caffarelli, L. A., *Elliptic second order equations*, Rendiconti del Seminario Matematico e Fisico di Milano, Vol LVIII (1988), 253-284.

[C3] Caffarelli, L. A., *A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity*, Annals of Mathematics 131 (1990), 129-134.

[C4] Caffarelli, L. A., *Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation*, Annals of Mathematics 131 (1990), 135-150.

[CNS] Caffarelli, L. A., Nirenberg, L., Spruck, J., *The Dirichlet problem for nonlinear second order elliptic equations I. Monge-Ampère equation*, Comm. Pure Appl. Math. 37 (1984), 369-402.

[CKNS] Caffarelli, L. A., Kohn, J. J., Nirenberg, L., Spruck, J., *The Dirichlet problem for nonlinear second order elliptic equations II*, Comm. Pure Appl. Math. 38 (1985), 209-252.

[Cm] Campanato, S., *Equazione ellittiche del secondo ordine e spazi $\mathcal{L}^{2,\lambda}$*, Ann. Mat. Pura Appl. 69 (1965), 321-380.

[CL] Crandall, M. G., Lions, P. L., *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. 277 (1983), 1-42.

[E1] Evans, L. C., *A convergence theorem for solutions of nonlinear second-order elliptic equations*, Indiana Univ. Math. J. 27 (1978), 875-887.

[E2] Evans, L. C., *On solving certain nonlinear partial differential equations by accretive operator methods*, Israel J. Math. 36 (1980), 225-247.

[E3] Evans, L. C., *Classical solutions of fully nonlinear, convex, second-order elliptic equations*, Comm. Pure Appl. Math. XXV (1982), 333-363.

[EG] Evans, L. C., Gariepy, R. F., *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, 1992.

[Es] Escauriaza, L., *$W^{2,n}$ a priori estimates for solutions to fully non-linear equations*, Indiana Univ. Math. J. 42, no 2 (1993), 413-423.

[F] Federer, H., *Geometric Measure Theory*, Springer-Verlag, 1969.

[G] Giaquinta, M., *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Princeton University Press, 1983.

[GT] Gilbarg, D., Trudinger, N. S. *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer-Verlag, 1983.

[dG] Guzmán, M. de, *Differentiation of Integrals in* $\mathbb{R}^n$, Lecture Notes in Math. 481, Springer-Verlag, 1975.

[I1] Ishii, H., *On uniqueness and existence of viscosity solutions of fully nonlinear second order elliptic PDE's*, Comm. Pure Appl. Math. XLII, no 1 (1989), 15-46.

[I2] Ishii, H., *Perron's method for Hamilton-Jacobi equations*, Duke Math. J. 55 (1987) no 2, 369-384.

[J] Jensen, R., *The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations*, Arch. Rational Mech. Anal. 101 (1988), no 1, 1-27.

[JLS] Jensen, R., Lions, P. L., Souganidis. P. E., *A uniqueness result for viscosity solutions of fully nonlinear second order partial differential equations*, Proc. Amer. Math. Soc. 4 (1988), 975-978.

[Ka] Kazdan, J. L., *Prescribing the Curvature of a Riemannian Manifold*, Regional conference series in mathematics 57, 1984.

[K1] Krylov, N. V., *Nonlinear Elliptic and Parabolic Equations of the Second Order*, Mathematics and its Applications, Reidel, 1987.

[K2] Krylov, N. V., *Controlled Difusion Processes*, Springer-Verlag, 1980.

[K3] Krylov, N. V., *Boundedly nonhomogeneous elliptic and parabolic equations*, Izv. Akad. Nak. SSSR Ser. Mat. 46 (1982), 487-523; English transl. in Math. USSR Izv. 20 (1983), 459-492.

[K4] Krylov, N. V., *Boundedly nonhomogeneous elliptic and parabolic equations in a domain*, Izv. Akad. Nak. SSSR Ser. Mat. 47 (1983), 75-108; English transl. in Math. USSR Izv. 22 (1984), 67-97.

[KS1] Krylov, N. V., Safonov, M. V., *An estimate of the probability that a diffusion process hits a set of positive measure*, Dokl. Akad. Nauk. SSSR 245 (1979), 253-255; English translation in Soviet Math. Dokl. 20 (1979), 253-255.

[KS2] Krylov, N. V., Safonov, M. V., *Certain properties of solutions of parabolic equations with measurable coefficients*, Izvestia Akad. Nauk. SSSR 40 (1980), 161-175.

[L] Lin, F. H., *Second derivative* L^p*-estimate for elliptic equations of nondivergent type*, Proceedings of the Amer. Math. Soc. Vol 96, no 3 (March 1986), 447-451.

[N] Nisio, M., *Stochastic differential games and viscosity solutions of Isaacs equations*, Nagoya Math. J. 110 (1988), 163-184.

[R] Rudin, W., *Functional Analysis*, McGraw-Hill, 1973.

[S1] Safonov, M. V., *On the classical solution of Bellman's elliptic equation*, Soviet Math. Dokl. 30 (1984), 482-485.

[S2] Safonov, M. V., *On the classical solution of nonlinear elliptic equations of second order*, Math. USSR Izvestiya Vol 33, no 3 (1989), 597-612.

[Sc] Schneider, R., *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge Univ. Press, 1993.

[T1] Trudinger, N. S., *Hölder gradient estimates for fully nonlinear equations*, Proc. Roy. Soc. Edinburgh, Sect. A 108 (1988), 57-65.

[T2] Trudinger, N. S., *Comparison principles and pointwise estimates for viscosity solutions of second order elliptic equations*, Rev. Mat. Iberoamericana, Vol 4, no 3 and 4 (1988), 453-468.

[T3] Trudinger, N. S., *On regularity and existence of viscosity solutions of nonlinear seeond order elliptic equations*, Essays of Math. Analysis in honour of E. De Giorgi for his sixtieth birthday, Birkhausser (1989), 939-957.

[W] Wang, L., *On the regularity theory of fully nonlinear parabolic equations I*, Comm. Pure Appl. Math. 45 (1992), no 1, 27-76, and *On the regularity theory of fully nonlinear parabolic equations II*, Comm. Pure Appl. Math. 45 (1992), no 2, 141-178.

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The goal of the book is to extend classical regularity theorems for solutions of linear elliptic partial differential equations to the context of fully nonlinear elliptic equations. This class of equations often arises in control theory, optimization, and other applications. The authors give a detailed presentation of all the necessary techniques. Instead of treating these techniques in their greatest generality, they outline the key ideas and prove the results needed for developing the subsequent theory.

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