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Fully Nonlinear Elliptic Equations

Luis A. Caffarelli
Xavier Cabré

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**Luis A. Caffarelli
Xavier Cabré**



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ABSTRACT. This book provides a self-contained development of the regularity theory of solutions of fully nonlinear elliptic equations. These partial differential equations arise in control theory and optimization. The goal of this work is to extend the classical Schauder and Calderón-Zygmund regularity theories for linear elliptic equations to the fully nonlinear context. The book contains a detailed presentation of all the techniques needed. We do not treat them in their greatest generality; rather we present the key ideas and prove all the results needed for the subsequent theory.

We develop the theory of viscosity solutions of nonlinear equations, the Alexandroff estimate and Krylov-Safonov Harnack inequality for viscosity solutions, Jensen's uniqueness theory for viscosity solutions, Evans and Krylov regularity theory for convex fully nonlinear equations, and finally the regularity theory for fully nonlinear equations with variable coefficients.

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The goal of the book is to extend classical regularity theorems for solutions of linear elliptic partial differential equations to the context of fully nonlinear elliptic equations. This class of equations often arises in control theory, optimization, and other applications. The authors give a detailed presentation of all the necessary techniques. Instead of treating these techniques in their greatest generality, they outline the key ideas and prove the results needed for developing the subsequent theory.

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