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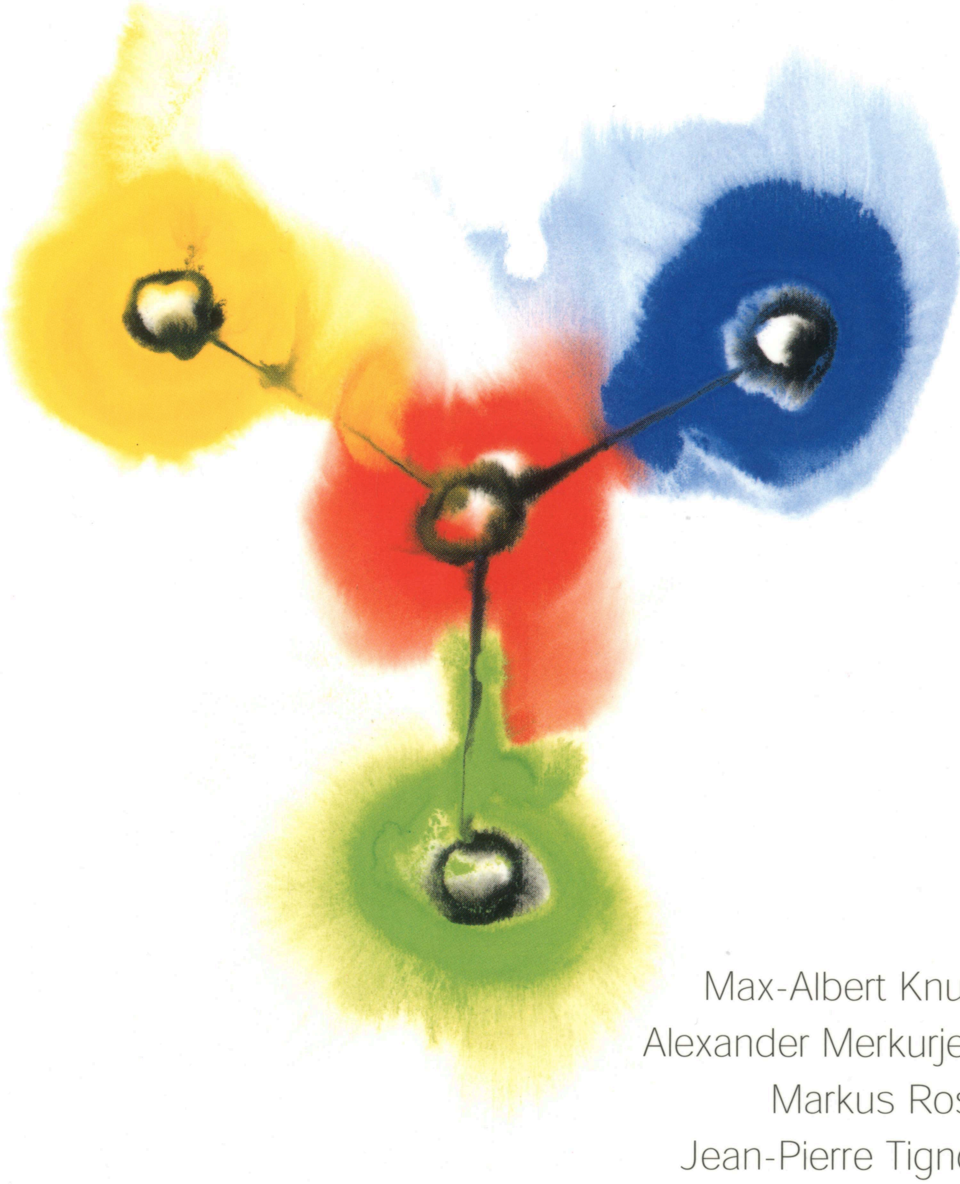
The Book of Involutions



Max-Albert Knus
Alexander Merkurjev
Markus Rost
Jean-Pierre Tignol

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1991 *Mathematics Subject Classification.* Primary 11E39, 11E57, 11E72; Secondary 11E88, 16K20, 16W10, 17A75, 17C40, 20G15.

This monograph yields a comprehensive exposition of the theory of central simple algebras with involution, in relation with linear algebraic groups. It aims to provide the algebra-theoretic foundations for much of the recent work on linear algebraic groups over arbitrary fields. Involutions are viewed as twisted forms of similarity classes of hermitian or bilinear forms, leading to new developments on the model of the algebraic theory of quadratic forms. Besides classical groups, phenomena related to triality are also discussed, as well as groups of type F_4 or G_2 arising from exceptional Jordan or composition algebras. Several results and notions appear here for the first time, notably the discriminant algebra of an algebra with unitary involution and the algebra-theoretic counterpart to linear groups of type D_4.

For research mathematicians and graduate students working in central simple algebras, algebraic groups, nonabelian Galois cohomology or Jordan algebras.

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Volume 44

The Book of Involutions

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Markus Rost

Jean-Pierre Tignol

With a preface by J. Tits

Contents

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Préface

Quatre des meilleurs algébristes d'aujourd'hui (j'aimerais dire, comme jadis, ≪géomètres≫, au sens noble mais hélas désuet du terme) nous donnent ce beau *Livre des Involutions*, qu'ils me demandent de préfacer.

Quel est le propos de l'ouvrage et à quels lecteurs s'adresse-t-il? Bien sûr il y est souvent question d'involutions, mais celles-ci sont loin d'être omniprésentes et le titre est plus l'expression d'un état d'âme que l'affirmation d'un thème central. En fait, les questions envisagées sont multiples, relevant toutes de domaines importants des mathématiques contemporaines ; sans vouloir être exhaustif (ceci n'est pas une introduction), on peut citer :

- les formes quadratiques et les algèbres de Clifford,
- les algèbres associatives centrales simples (ici les involutions, et notamment celles de seconde espèce, se taillent une place de choix !) mais aussi les algèbres alternatives et les algèbres de Jordan,
- les algèbres de Hopf,
- les groupes algébriques, principalement semi-simples,
- la cohomologie galoisienne.

Pour ce qui est du public concerné, la lecture ou la consultation du livre sera profitable à un large éventail de mathématiciens. Le non-initié y trouvera une introduction claire aux concepts fondamentaux des domaines en question ; exposés le plus souvent en fonction d'applications concrètes, ces notions de base sont présentées de façon vivante et dépouillée, sans généralités gratuites (les auteurs ne sont pas adeptes de grandes théories abstraites). Le lecteur déjà informé, ou croyant l'être, pourra réapprendre (ou découvrir) quelques beaux théorèmes jadis ≪bien connus≫ mais un peu oubliés dans la littérature récente, ou au contraire, voir des résultats qui lui sont en principe familiers exposés sous un jour nouveau et éclairant (je pense par exemple à l'introduction des algèbres trialitaires au dernier chapitre). Enfin, les spécialistes et les chercheurs auront à leur disposition une référence précieuse, parfois unique, pour des développements récents, souvents dûs aux auteurs eux-mêmes, et dont certains sont exposés ici pour la première fois (c'est par exemple le cas pour plusieurs résultats sur les invariants cohomologiques, donnés à la fin du chapitre 7).

Malgré la grande variété des thèmes considérés et les individualités très marquées des quatre auteurs, ce Livre des Involutions a une unité remarquable. Le ciment un peu fragile des involutions n'est certes pas seul à l'expliquer. Il y a aussi, bien sûr, les interconnections multiples entre les sujets traités ; mais plus déterminante encore est l'importance primordiale accordée à des structures fortes, se prêtant par exemple à des théorèmes de classification substantiels. Ce n'est pas un hasard si les algèbres centrales simples de petites dimensions (trois et quatre), les groupes exceptionnels de type G_2 et F_4 (on regrette un peu que Sa Majesté E_8

fasse ici figure de parent pauvre), les algèbres de composition, ..., reçoivent autant d'attention.

On l'a compris, ce Livre est tout à la fois un livre de lecture passionnant et un ouvrage de référence d'une extrême richesse. Je suis reconnaissant aux auteurs de l'honneur qu'ils m'ont fait en me demandant de le préfacer, et plus encore de m'avoir permis de le découvrir et d'apprendre à m'en servir.

Jacques Tits

Introduction

For us an involution is an anti-automorphism of order two of an algebra. The most elementary example is the transpose for matrix algebras. A more complicated example of an algebra over $\mathbb{Q}$ admitting an involution is the multiplication algebra of a Riemann surface (see the notes at the end of Chapter I for more details). The central problem here, to give necessary and sufficient conditions on a division algebra over $\mathbb{Q}$ to be a multiplication algebra, was completely solved by Albert (1934/35). To achieve this, Albert developed a theory of central simple algebras with involution, based on the theory of simple algebras initiated a few years earlier by Brauer, Noether, and also Albert and Hasse, and gave a complete classification over $\mathbb{Q}$. This is the historical origin of our subject, however our motivation has a different source. The basic objects are still central simple algebras, i.e., "forms" of matrix algebras. As observed by Weil (1960), central simple algebras with involution occur in relation to classical algebraic simple adjoint groups: connected components of automorphism groups of central simple algebras with involution are such groups (with the exception of a quaternion algebra with an orthogonal involution, where the connected component of the automorphism group is a torus), and, in their turn, such groups are connected components of automorphism groups of central simple algebras with involution.

Even if this is mainly a book on algebras, the correspondence between algebras and groups is a constant leitmotiv. Properties of the algebras are reflected in properties of the groups and of related structures, such as Dynkin diagrams, and vice versa. For example we associate certain algebras to algebras with involution in a functorial way, such as the Clifford algebra (for orthogonal involutions) or the λ-powers and the discriminant algebra (for unitary involutions). These algebras are exactly the "Tits algebras," defined by Tits (1971) in terms of irreducible representations of the groups. Another example is algebraic triality, which is historically related with groups of type D_4 (É. Cartan) and whose "algebra" counterpart is, so far as we know, systematically approached here for the first time.

In the first chapter we recall basic properties of central simple algebras and involutions. As a rule for the whole book, without however going to the utmost limit, we try to allow base fields of characteristic 2 as well as those of other characteristic. Involutions are divided up into orthogonal, symplectic and unitary types. A central idea of this chapter is to interpret involutions in terms of hermitian forms over skew fields. Quadratic pairs, introduced at the end of the chapter, give a corresponding interpretation for quadratic forms in characteristic 2.

In Chapter II we define several invariants of involutions; the index is defined for every type of involution. For quadratic pairs additional invariants are the discriminant, the (even) Clifford algebra and the Clifford module; for unitary involutions we introduce the discriminant algebra. The definition of the discriminant algebra is prepared for by the construction of the λ-powers of a central simple algebra. The last part of this chapter is devoted to trace forms on algebras, which represent an important tool for recent results discussed in later parts of the book. Our method of definition is based on scalar extension: after specifying the definitions "rationally" (i.e., over an arbitrary base field), the main properties are proven by working over a splitting field. This is in contrast to Galois descent, where constructions over a separable closure are shown to be invariant under the Galois group and therefore are defined over the base field. A main source of inspiration for Chapters I and II is the paper [291] of Tits on "Formes quadratiques, groupes orthogonaux et algèbres de Clifford."

In Chapter III we investigate the automorphism groups of central simple algebras with involutions. Inner automorphisms are induced by elements which we call similitudes. These automorphism groups are twisted forms of the classical projective orthogonal, symplectic and unitary groups. After proving results which hold for all types of involutions, we focus on orthogonal and unitary involutions, where additional information can be derived from the invariants defined in Chapter II. The next two chapters are devoted to algebras of low degree. There exist certain isomorphisms among classical groups, known as exceptional isomorphisms. From the algebra point of view, this is explained in the first part of Chapter IV by properties of the Clifford algebra of orthogonal involutions on algebras of degree 3, 4, 5 and 6. In the second part we focus on tensor products of two quaternion algebras, which we call biquaternion algebras. These algebras have many interesting properties, which could be the subject of a monograph of its own. This idea was at the origin of our project.

Algebras with unitary involutions are also of interest for odd degrees, the lowest case being degree 3. From the group point of view algebras with unitary involutions of degree 3 are of type A_2. Chapter V gives a new presentation of results of Albert and a complete classification of these algebras. In preparation for this, we recall general results on étale and Galois algebras.

The aim of Chapter VI is to give the classification of semisimple algebraic groups over arbitrary fields. We use the functorial approach to algebraic groups, although we quote without proof some basic results on algebraic groups over algebraically closed fields. In the central section we describe in detail Weil's correspondence [310] between central simple algebras with involution and classical groups. Exceptional isomorphisms are reviewed again in terms of this correspondence. In the last section we define Tits algebras of semisimple groups and give explicit constructions of them in classical cases.

The theme of Chapter VII is Galois cohomology. We introduce the formalism and describe many examples. Previous results are reinterpreted in this setting and cohomological invariants are discussed. Most of the techniques developed here are also needed for the following chapters.

The last three chapters are dedicated to the exceptional groups of type G_2, F_4 and to D_4, which, in view of triality, is also exceptional. In the Weil correspondence, octonion algebras play the algebra role for G_2 and exceptional simple Jordan algebras the algebra role for F_4.

Octonion algebras are an important class of composition algebras and Chapter VIII gives an extensive discussion of composition algebras. Of special interest from the group point of view are "symmetric" compositions. In dimension 8 these are of two types, corresponding to algebraic groups of type A_2 or type G_2. Triality is defined through the Clifford algebra of symmetric 8-dimensional compositions. As a step towards exceptional simple Jordan algebras, we introduce twisted compositions, which are defined over cubic étale algebras. This generalizes a construction of Springer. The corresponding group of automorphisms in the split case is the semidirect product $\mathrm{Spin}_8 \rtimes S_3$.

In Chapter IX we describe different constructions of exceptional simple Jordan algebras, due to Freudenthal, Springer and Tits (the algebra side) and give interpretations from the algebraic group side. The Springer construction arises from twisted compositions, defined in Chapter VIII, and basic ingredients of Tits constructions are algebras of degree 3 with unitary involutions, studied in Chapter III. We conclude this chapter by defining cohomological invariants for exceptional simple Jordan algebras.

The last chapter deals with trialitarian actions on simple adjoint groups of type D_4. To complete Weil's program for outer forms of D_4 (a case not treated by Weil), we introduce a new notion, which we call a trialitarian algebra. The underlying structure is a central simple algebra with an orthogonal involution, of degree 8 over a cubic étale algebra. The trialitarian condition relates the algebra to its Clifford algebra. Trialitarian algebras also occur in the construction of Lie algebras of type D_4. Some indications in this direction are given in the last section.

Exercises and notes can be found at the end of each chapter. Omitted proofs sometimes occur as exercises. Moreover we included as exercises some results we like, but which we did not wish to develop fully. In the notes we wanted to give complements and to look at some results from a historical perspective. We have tried our best to be useful; we cannot, however, give strong guarantees of completeness or even fairness.

This book is the achievement of a joint (and very exciting) effort of four very different people. We are aware that the result is still quite heterogeneous; however, we flatter ourselves that the differences in style may be viewed as a positive feature.

Our work started out as an attempt to understand Tits' definition of the Clifford algebra of a generalized quadratic form, and ended up including many other topics to which Tits made fundamental contributions, such as linear algebraic groups, exceptional algebras, triality, ... Not only was Jacques Tits a constant source of inspiration through his work, but he also had a direct personal influence, notably through his threat — early in the inception of our project — to speak evil of our work if it did not include the characteristic 2 case. Finally he also agreed to bestow his blessings on our book sous forme de préface. For all that we thank him wholeheartedly.

This book could not have been written without the help and the encouragement of many friends. They are too numerous to be listed here individually, but we hope they will recognize themselves and find here our warmest thanks. Richard Elman deserves a special mention for his comment that the most useful book is not the one to which nothing can be added, but the one which is published. This no-nonsense statement helped us set limits to our endeavor. We were fortunate to get useful advice on various points of the exposition from Ottmar Loos, Antonio Paques, Parimala, Michel Racine, David Saltman, Jean-Pierre Serre and Sridharan. We thank all of them for lending helping hands at the right time. A number of people were nice enough to read and comment on drafts of parts of this book: Eva Bayer-Fluckiger, Vladimir Chernousov, Ingrid Dejaiffe, Alberto Elduque, Darrell Haile, Luc Haine, Pat Morandi, Holger Petersson, Ahmed Serhir, Tony Springer, Paul Swets and Oliver Villa. We know all of them had better things to do, and we are grateful. Skip Garibaldi and Adrian Wadsworth actually summoned enough grim self-discipline to read a draft of the whole book, detecting many shortcomings, making shrewd comments on the organization of the book and polishing our broken English. Each deserves a medal. However, our capacity for making mistakes certainly exceeds our friends' sagacity. We shall gratefully welcome any comment or correction.

Tignol had the privilege to give a series of lectures on "Central simple algebras, involutions and quadratic forms" in April 1993 at the National Taiwan University. He wants to thank Ming-chang Kang and the National Research Council of China for this opportunity to test high doses of involutions on a very patient audience, and Eng-Tjioe Tan for making his stay in Taiwan a most pleasant experience. The lecture notes from this crash course served as a blueprint for the first chapters of this book.

Our project immensely benefited by reciprocal visits among the authors. We should like to mention with particular gratitude Merkurjev's stay in Louvain-la-Neuve in 1993, with support from the Fonds de Développement Scientifique and the Institut de Mathématique Pure et Appliquée of the Université catholique de Louvain, Rost's stays in Zürich at the Mathematics Research Institute (FIM) in winter 1995–96 and at the Mathematics Department of the ETH Zürich for the winter semester of 1996–97, and Tignol's stay at the Mathematics Department of the ETH Zürich for the winter semester of 1995–96, both with support from the Eidgenössische Technische Hochschule. Moreover, Merkurjev gratefully acknowledges support from the Alexander von Humboldt foundation and the hospitality of the Bielefeld university for the year 1995–96, and Tignol is grateful to the National Fund for Scientific Research of Belgium and to the European Commission for partial support under the "crédit aux chercheurs" and the TMR programmes respectively (contract ERB-FMRX-CT97-0107).

The four authors enthusiastically thank Herbert Rost (Markus' father) for the design of the cover page, in particular for his wonderful and colorful rendition of the Dynkin diagram D_4. They also give special praise to Sergei Gelfand, Director of Acquisitions of the American Mathematical Society, for his helpfulness and patience in taking care of all their wishes for the publication.

Conventions and Notations

Maps. The image of an element x under a map f is generally denoted $f(x)$; the notation x^f is also used however, notably for homomorphisms of left modules. In that case, we also use the right-hand rule for mapping composition; for the image of $x \in X$ under the composite map $X \xrightarrow{f} Y \xrightarrow{g} Z$ we set either $g \circ f(x)$ or x^{fg} and the composite is thus either $g \circ f$ or fg.

As a general rule, module homomorphisms are written on the opposite side of the scalars. (Right modules are usually preferred.) Thus, if M is a module over a ring R, it is also a module (on the opposite side) over $\operatorname{End}_R(M)$, and the R-module structure defines a natural homomorphism:

$$R \to \operatorname{End}_{\operatorname{End}_R(M)}(M).$$

Note therefore that if $S \subset \operatorname{End}_R(M)$ is a subring, and if we endow M with its natural S-module structure, then $\operatorname{End}_S(M)$ is the *opposite* of the centralizer of S in $\operatorname{End}_R(M)$:

$$\operatorname{End}_S(M) = \left(C_{\operatorname{End}_R(M)}S\right)^{\mathrm{op}}.$$

Of course, if R is commutative, every right R-module M_R may also be regarded as a left R-module ${}_RM$, and every endomorphism of M_R also is an endomorphism of ${}_RM$. Note however that with the convention above, the canonical map $\operatorname{End}_R(M_R) \to \operatorname{End}_R({}_RM)$ is an anti-isomorphism.

The characteristic polynomial and its coefficients. Let F denote an arbitrary field. The characteristic polynomial of a matrix $m \in M_n(F)$ (or an endomorphism m of an n-dimensional F-vector space) is denoted

(0.1) $$P_m(X) = X^n - s_1(m)X^{n-1} + s_2(m)X^{n-2} - \cdots + (-1)^n s_n(m).$$

The trace and determinant of m are denoted $\operatorname{tr}(m)$ and $\det(m)$:

$$\operatorname{tr}(m) = s_1(m), \qquad \det(m) = s_n(m).$$

We recall the following relations between coefficients of the characteristic polynomial:

(0.2) Proposition. *For m, $m' \in M_n(F)$, we have $s_1(m)^2 - s_1(m^2) = 2s_2(m)$ and*

$$s_1(m)s_1(m') - s_1(mm') = s_2(m+m') - s_2(m) - s_2(m').$$

Proof: It suffices to prove these relations for generic matrices $m = (x_{ij})_{1\le i,j\le n}$, $m' = (x'_{ij})_{1\le i,j\le n}$ whose entries are indeterminates over $\mathbb{Z}$; the general case follows by specialization. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the generic matrix m (in an algebraic closure of $\mathbb{Q}(x_{ij} \mid 1 \leq i,j \leq n)$), we have $s_1(m) = \sum_{1\leq i\leq n} \lambda_i$ and $s_2(m) = \sum_{1\leq i<j\leq n} \lambda_i\lambda_j$, hence

$$s_1(m)^2 - 2s_2(m) = \sum_{1\leq i\leq n} \lambda_i^2 = s_1(m^2),$$

proving the first relation. The second relation follows by linearization, since 2 is not a zero-divisor in $\mathbb{Z}[x_{ij}, x'_{ij} \mid 1 \leq i,j \leq n]$. □

If L is an associative and commutative F-algebra of dimension n and $\ell \in L$, the characteristic polynomial of multiplication by ℓ, viewed as an F-endomorphism of L, is called the *generic polynomial of* ℓ and is denoted

$$P_{L,\ell}(X) = X^n - s_1(\ell)X^{n-1} + s_2(\ell)X^{n-2} - \cdots + (-1)^n s_n(\ell).$$

The trace and norm of ℓ are denoted $T_{L/F}(\ell)$ and $N_{L/F}(\ell)$ (or simply $T(\ell)$, $N(\ell)$):

$$T_{L/F}(\ell) = s_1(\ell), \qquad N_{L/F}(\ell) = s_n(\ell).$$

We also denote

(0.3) $$S_{L/F}(\ell) = S(\ell) = s_2(\ell).$$

The characteristic polynomial is also used to define a generic polynomial for central simple algebras, called the *reduced characteristic polynomial*: see (1.6). Generalizations to certain nonassociative algebras are given in § 32.

Bilinear forms. A bilinear form $b\colon V \times V \to F$ on a finite dimensional vector space V over an arbitrary field F is called *symmetric* if $b(x,y) = b(y,x)$ for all x, $y \in V$, *skew-symmetric* if $b(x,y) = -b(y,x)$ for all x, $y \in V$ and *alternating* if $b(x,x) = 0$ for all $x \in V$. Thus, the notions of skew-symmetric and alternating (resp. symmetric) form coincide if $\operatorname{char} F \neq 2$ (resp. $\operatorname{char} F = 2$). Alternating forms are skew-symmetric in every characteristic.

If b is a symmetric or alternating bilinear form on a (finite dimensional) vector space V, the induced map

$$\hat{b}\colon V \to V^* = \operatorname{Hom}_F(V,F)$$

is defined by $\hat{b}(x)(y) = b(x,y)$ for x, $y \in V$. The bilinear form b is *nonsingular* (or *regular*, or *nondegenerate*) if $\hat{b}$ is bijective. (It suffices to require that $\hat{b}$ be injective, i.e., that the only vector $x \in V$ such that $b(x,y) = 0$ for all $y \in V$ is $x = 0$, since we are dealing with finite dimensional vector spaces over fields.) Alternately, b is nonsingular if and only if the determinant of its Gram matrix with respect to an arbitrary basis of V is nonzero:

$$\det\bigl(b(e_i,e_j)\bigr)_{1\leq i,j\leq n} \neq 0.$$

In that case, the square class of this determinant is called the *determinant* of b :

$$\det b = \det\bigl(b(e_i,e_j)\bigr)_{1\leq i,j\leq n} \cdot F^{\times 2} \in F^\times/F^{\times 2}.$$

The *discriminant* of b is the signed determinant:

$$\operatorname{disc} b = (-1)^{n(n-1)/2} \det b \in F^\times/F^{\times 2} \quad \text{where } n = \dim V.$$

For $\alpha_1, \ldots, \alpha_n \in F$, the bilinear form $\langle \alpha_1, \ldots, \alpha_n\rangle$ on F^n is defined by

$$\langle \alpha_1, \ldots, \alpha_n\rangle\bigl((x_1,\ldots,x_n),(y_1,\ldots,y_n)\bigr) = \alpha_1 x_1 y_1 + \cdots + \alpha_n x_n y_n.$$

We also define the *n-fold Pfister bilinear form* $\langle\langle \alpha_1, \ldots, \alpha_n\rangle\rangle$ by

$$\langle\langle \alpha_1, \ldots, \alpha_n\rangle\rangle = \langle 1, -\alpha_1\rangle \otimes \cdots \otimes \langle 1, -\alpha_n\rangle.$$

If $b\colon V\times V \to F$ is a symmetric bilinear form, we denote by $q_b\colon V \to F$ the associated quadratic map, defined by

$$q_b(x) = b(x,x) \qquad \text{for } x \in V.$$

Quadratic forms. If $q\colon V \to F$ is a quadratic map on a finite dimensional vector space over an arbitrary field F, the associated symmetric bilinear form b_q is called the *polar* of q; it is defined by

$$b_q(x,y) = q(x+y) - q(x) - q(y) \qquad \text{for } x,\ y \in V,$$

hence $b_q(x,x) = 2q(x)$ for all $x \in V$. Thus, the quadratic map q_{b_q} associated to b_q is $q_{b_q} = 2q$. Similarly, for every symmetric bilinear form b on V, we have $b_{q_b} = 2b$.

Let $V^\perp = \{\, x \in V \mid b_q(x,y) = 0 \text{ for } y \in V\,\}$. The quadratic map q is called *nonsingular* (or *regular*, or *nondegenerate*) if either $V^\perp = \{0\}$ or $\dim V^\perp = 1$ and $q(V^\perp) \neq \{0\}$. The latter case occurs only if $\operatorname{char} F = 2$ and V is odd-dimensional. Equivalently, a quadratic form of dimension n is nonsingular if and only if it is equivalent over an algebraic closure to $\sum_{i=1}^{n/2} x_{2i-1}x_{2i}$ (if n is even) or to $x_0^2 + \sum_{i=1}^{(n-1)/2} x_{2i-1}x_{2i}$ (if n is odd).

The *determinant* and the *discriminant* of a nonsingular quadratic form q of dimension n over a field F are defined as follows: let M be a matrix representing q in the sense that

$$q(X) = X \cdot M \cdot X^t$$

where $X = (x_1, \dots, x_n)$ and ^{t} denotes the transpose of matrices; the condition that q is nonsingular implies that $M + M^t$ is invertible if n is even or $\operatorname{char} F \neq 2$, and has rank $n-1$ if n is odd and $\operatorname{char} F = 2$. The matrix M is uniquely determined by q up to the addition of a matrix of the form $N - N^t$; therefore, $M + M^t$ is uniquely determined by q.

If $\operatorname{char} F \neq 2$ we set

$$\det q = \det\bigl(\tfrac{1}{2}(M + M^t)\bigr) \cdot F^{\times 2} \in F^\times/F^{\times 2}$$

and

$$\operatorname{disc} q = (-1)^{n(n-1)/2} \det q \in F^\times/F^{\times 2}.$$

Thus, the determinant (resp. the discriminant) of a quadratic form is the determinant (resp. the discriminant) of its polar form divided by 2^n.

If $\operatorname{char} F = 2$ and n is odd we set

(0.4) $$\det q = \operatorname{disc} q = q(y) \cdot F^{\times 2} \in F^\times/F^{\times 2}$$

where $y \in F^n$ is a nonzero vector such that $(M + M^t) \cdot y = 0$. Such a vector y is uniquely determined up to a scalar factor, since $M + M^t$ has rank $n-1$, hence the definition above does not depend on the choice of y.

If $\operatorname{char} F = 2$ and n is even we set

$$\det q = s_2\bigl((M + M^t)^{-1}M\bigr) + \wp(F) \in F/\wp(F)$$

and

$$\operatorname{disc} q = \tfrac{m(m-1)}{2} + \det q \in F/\wp(F)$$

where $m = n/2$ and $\wp(F) = \{x + x^2 \mid x \in F\}$. (More generally, for fields of characteristic $p \neq 0$, $\wp$ is defined as $\wp(x) = x + x^p$, $x \in F$.) The following lemma shows that the definition of $\det q$ does not depend on the choice of M:

(0.5) Lemma. *Suppose* $\operatorname{char} F = 2$. *Let* $M, N \in M_n(F)$ *and* $W = M + M^t$. *If* W *is invertible, then*

$$s_2\big(W^{-1}(M + N + N^t)\big) = s_2(W^{-1}M) + s_1(W^{-1}N) + \big(s_1(W^{-1}N)\big)^2.$$

Proof: The second relation in (0.2) yields

$$\begin{aligned} s_2\big(W^{-1}M + W^{-1}(N + N^t)\big) = \\ s_2(W^{-1}M) + s_2\big(W^{-1}(N + N^t)\big) + s_1(W^{-1}M)s_1\big(W^{-1}(N + N^t)\big) \\ + s_1\big(W^{-1}MW^{-1}(N + N^t)\big). \end{aligned}$$

In order to prove the lemma, we show below:

$$s_2\big(W^{-1}(N + N^t)\big) = \big(s_1(W^{-1}N)\big)^2 \tag{0.6}$$

$$s_1(W^{-1}M)s_1\big(W^{-1}(N + N^t)\big) = 0 \tag{0.7}$$

$$s_1\big(W^{-1}MW^{-1}(N + N^t)\big) = s_1(W^{-1}N). \tag{0.8}$$

Since a matrix and its transpose have the same characteristic polynomial, the traces of $W^{-1}N$ and $(W^{-1}N)^t = N^tW^{-1}$ are the same, hence

$$s_1(W^{-1}N^t) = s_1(N^tW^{-1}) = s_1(W^{-1}N).$$

Therefore, $s_1\big(W^{-1}(N + N^t)\big) = 0$, and (0.7) follows.

Similarly, we have

$$s_1(W^{-1}MW^{-1}N^t) = s_1(NW^{-1}M^tW^{-1}) = s_1(W^{-1}M^tW^{-1}N),$$

hence the left side of (0.8) is

$$s_1(W^{-1}MW^{-1}N) + s_1(W^{-1}M^tW^{-1}N) = s_1\big(W^{-1}(M + M^t)W^{-1}N\big).$$

Since $M + M^t = W$, (0.8) follows.

The second relation in (0.2) shows that the left side of (0.6) is

$$s_2(W^{-1}N) + s_2(W^{-1}N^t) + s_1(W^{-1}N)s_1(W^{-1}N^t) + s_1(W^{-1}NW^{-1}N^t).$$

Since $W^{-1}N$ and $W^{-1}(W^{-1}N)^tW$ $(= W^{-1}N^t)$ have the same characteristic polynomial, we have $s_i(W^{-1}N) = s_i(W^{-1}N^t)$ for $i = 1, 2$, hence the first two terms cancel and the third is equal to $s_1(W^{-1}N)^2$. In order to prove (0.6), it therefore suffices to show

$$s_1(W^{-1}NW^{-1}N^t) = 0.$$

Since $W = M + M^t$, we have $W^{-1} = W^{-1}MW^{-1} + W^{-1}M^tW^{-1}$, hence

$$s_1(W^{-1}NW^{-1}N^t) = s_1(W^{-1}MW^{-1}NW^{-1}N^t) + s_1(W^{-1}M^tW^{-1}NW^{-1}N^t),$$

and (0.6) follows if we show that the two terms on the right side are equal. Since $W^t = W$ we have $(W^{-1}MW^{-1}NW^{-1}N^t)^t = NW^{-1}N^tW^{-1}M^tW^{-1}$, hence

$$\begin{aligned} s_1(W^{-1}MW^{-1}NW^{-1}N^t) &= s_1\big((NW^{-1}N^t)(W^{-1}M^tW^{-1})\big) \\ &= s_1(W^{-1}M^tW^{-1}NW^{-1}N^t). \end{aligned}$$

□

Quadratic forms are called *equivalent* if they can be transformed into each other by invertible linear changes of variables. The various quadratic forms representing a quadratic map with respect to various bases are thus equivalent. It is easily verified that the determinant $\det q$ (hence also the discriminant $\operatorname{disc} q$) is an invariant of the equivalence class of the quadratic form q; the determinant and the discriminant are therefore also defined for quadratic maps. The discriminant of a quadratic form (or map) of even dimension in characteristic 2 is also known as the *pseudodiscriminant* or the *Arf invariant* of the form. See § 8.D for the relation between the discriminant and the even Clifford algebra.

Let $\alpha_1, \ldots, \alpha_n \in F$. If $\operatorname{char} F \neq 2$ we denote by $\langle\alpha_1, \ldots, \alpha_n\rangle$ the diagonal quadratic form

$$\langle\alpha_1, \ldots, \alpha_n\rangle = \alpha_1 x_1^2 + \cdots + \alpha_n x_n^2$$

which is the quadratic form associated to the bilinear form $\langle\alpha_1, \ldots, \alpha_n\rangle$. We also define the *n-fold Pfister quadratic form* $\langle\langle\alpha_1, \ldots, \alpha_n\rangle\rangle$ by

$$\langle\langle\alpha_1, \ldots, \alpha_n\rangle\rangle = \langle 1, -\alpha_1\rangle \otimes \cdots \otimes \langle 1, -\alpha_n\rangle$$

where $\otimes = \otimes_F$ is the tensor product over F. If $\operatorname{char} F = 2$, the quadratic forms $[\alpha_1, \alpha_2]$ and $[\alpha_1]$ are defined by

$$[\alpha_1, \alpha_2] = \alpha_1 X_1^2 + X_1 X_2 + \alpha_2 X_2^2 \qquad \text{and} \qquad [\alpha_1] = \alpha_1 X^2,$$

and the *n-fold Pfister quadratic form* $\langle\langle\alpha_1, \ldots, \alpha_n]]$ by

$$\langle\langle\alpha_1, \ldots, \alpha_n]] = \langle\langle\alpha_1, \ldots, \alpha_{n-1}\rangle\rangle \otimes [1, \alpha_n].$$

(See Baeza [28, p. 5] or Knus [157, p. 50] for the definition of the tensor product of a bilinear form and a quadratic form.) For instance,

$$\langle\langle\alpha_1, \alpha_2]] = (x_1^2 + x_1 x_2 + \alpha_2 x_2^2) + \alpha_1(x_3^2 + x_3 x_4 + \alpha_2 x_4^2).$$

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