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Random Matrices, Frobenius Eigenvalues, and Monodromy

Nicholas M. Katz

Peter Sarnak



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ABSTRACT. The main topic of this book is the deep relation between the spacings between zeros of zeta and L -functions and spacings between eigenvalues of random elements of large compact classical groups. This relation, the Montgomery-Odlyzko law, is shown to hold for wide classes of zeta and L -functions over finite fields.

The book draws on, and gives accessible accounts of, many disparate areas of mathematics, from algebraic geometry, moduli spaces, monodromy, equidistribution, and the Weil Conjectures, to probability theory on the compact classical groups in the limit as their dimension goes to infinity and related techniques from orthogonal polynomials and Fredholm determinants. It will be useful and interesting to researchers and graduate students working in any of these areas.

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References

- [A-K] Altman, A. and Kleiman, S., *Introduction to Grothendieck duality theory*, Lecture Notes in Math., vol. 146, Springer-Verlag, 1970.
- [Artin] Artin, E., *Quadratische Körper in Gebiete der höheren Kongruenzen*. I, II, Math. Z. **19** (1924), 153–246.
- [B-T-W] Basor, E., Tracy, C., and Widom, H., *Asymptotics of level-spacing distributions for random matrices*, Phys. Rev. Lett. **69** (1992), 5–8; Errata, Phys. Rev. Lett. **69** (1992), 2880.
- [Bott] Bott, R., *Homogeneous vector bundles*, Ann. of Math. (2) **66** (1957), 203–248.
- [Bour-L9] Bourbaki, N., *Groupes et algèbres de Lie*, Chapitre 9, Masson, Paris, 1982.
- [C-F] Chai, C. L. and Faltings, G., *Degeneration of abelian varieties*, Springer-Verlag, 1990.
- [Chav] Chavdarov, Nick, *The generic irreducibility of the numerator of the zeta function in a family of curves with large monodromy*, Duke Math. J. **87** (1997), 151–180.
- [De-AFT] Deligne, P., *Application de la formule des traces aux sommes trigonométriques*, Cohomologie Étale (SGA 4 1/2), Lecture Notes in Math., vol. 569, Springer-Verlag, 1977, pp. 168–232.
- [De-CCI] Deligne, P., *Cohomologie des intersections complètes*, Groupes de Monodromie en Géométrie Algébrique (SGA 7, Part II) Lecture Notes in Math., vol. 340, Springer-Verlag, 1973, pp. 39–61.
- [De-CEF] Deligne, P., *Courbes elliptiques: formulaire (d’après J. Tate)*, Modular Functions of One Variable. IV, Lecture Notes in Math., vol. 476, Springer-Verlag, 1975, pp. 57–73.
- [De-Mum] Deligne, P. and Mumford, D., *Irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 75–109.
- [De-Weil I] Deligne, P., *La conjecture de Weil*. I, Inst. Hautes Études Sci. Publ. Math. **48** (1974), 273–308.
- [De-Weil II] Deligne, P., *La conjecture de Weil*. II, Inst. Hautes Études Sci. Publ. Math. **52** (1981), 313–428.
- [Dw] Dwork, B., *On the rationality of the zeta function of an algebraic variety*, Amer. J. Math. **82** (1960), 632–648.
- [Fel] Feller, W., *An introduction to probability theory and its applications*. Vol. II, Wiley, 1966.
- [Fer] Fermigier, S., *Étude expérimentale du rang de familles de courbes elliptiques sur \mathbb{Q}* , Exper. Math. **5** (1966), 119–130.
- [FGA] Grothendieck, A., *Fondements de la géométrie algébrique* (a collection of his Bourbaki talks), Secrétariat Math., Paris, 1962.
- [Fuchs] Fuchs, W. H. J., *On the eigenvalues of an integral equation arising in the theory of band-limited signals*, J. Math. Anal. Appl. **9** (1964), 317–330.
- [Gaudin] Gaudin, M., *Sur la loi limite de l’espacement des valeurs propres d’une matrice aléatoire*, Nuclear Phys. **25** (1961), 447–458.
- [Gro-FL] Grothendieck, A., *Formule de Lefschetz et rationalité des fonctions L*, Séminaire Bourbaki 1964–65, Exposé 279, reprinted in *Dix exposés sur la cohomologie des schémas*, North-Holland, 1968.
- [Hasse] Hasse, H., *Zur Theorie der abstrakten elliptischen Funktionenkorper*. I, II, III, J. Reine Angew. Math. **175** (1936), 55–62, 69–88, 193–208.

- [Ig] Igusa, J., *Fibre systems of Jacobian varieties*. III, Amer. J. Math. **81** (1959), 561–577.
- [Ill-DFT] Illusie, L., *Deligne's l -adic Fourier transform*, Algebraic Geometry: Bowdoin 1985 (Bloch, S., ed.), Proc. Sympos. Pure Math., vol. 46, part 2, Amer. Math. Soc., 1987, pp. 151–163.
- [Ill-Ord] Illusie, L., *Ordinarité*, The Grothendieck Festschrift. Vol. II (P. Cartier et al., eds.), Birkhäuser, 1990, pp. 375–405.
- [Ka-ACT] Katz, N., *Affine cohomological transforms, perversity and monodromy*, J. Amer. Math. Soc. **6** (1993), 149–222.
- [Ka-BTBM] Katz, N., *Big twists have big monodromy*, in preparation.
- [Ka-ESDE] Katz, N., *Exponential sums and differential equations*, Ann. of Math. Studies, vol. 124, Princeton Univ. Press, 1990.
- [Ka-GKM] Katz, N., *Gauss sums, Kloosterman sums, and monodromy groups*, Ann. of Math. Studies, vol. 116, Princeton Univ. Press, 1988.
- [Ka-Lang] Katz, N., and Lang, S., *Finiteness theorems in geometric classfield theory*, L'Enseignement Math. (2) **27** (1981), 285–314.
- [Ka-Maz] Katz, N., and Mazur, B., *Arithmetic moduli of elliptic curves*, Ann. of Math. Studies, vol. 108, Princeton Univ. Press, 1985.
- [Ka-MG] Katz, N., *On the monodromy groups attached to certain families of exponential sums*, Duke Math. J. **54** (1987), 41–56.
- [Ka-ODW21] Katz, N., *An overview of Deligne's work on Hilbert's twenty-first problem*, Mathematical Developments Arising from Hilbert Problems, Proc. Sympos. Pure Math., vol. 28, Amer. Math. Soc., 1976, pp. 537–557.
- [Ka-RLS] Katz, N., *Rigid local systems*, Ann. of Math. Studies, vol. 138, Princeton Univ. Press, 1995.
- [Ka-Sar] Katz, N., and Sarnak, P., *Zeros of zeta functions and symmetry*, Bull. Amer. Math. Soc. (to appear).
- [Ka-SE] Katz, N., *Sommes exponentielles*, rédigé par G. Laumon, Astérisque, vol. 79, Soc. Math. France, 1980.
- [Ka-TA] Katz, N., *On a theorem of Ax*, Amer. J. Math. **93** (1971), 484–499.
- [Ka-TL] Katz, N., *Travaux de Laumon*, Séminaire Bourbaki 1987–88, Astérisque, vol. 161–162, Soc. Math. France, 1988, pp. 105–132.
- [K-S] Kodaira, K., and Spencer, D. C., *On deformations of complex structures*. II, Ann. of Math. (2) **67** (1958), 403–466.
- [Kra-Zag] Kramarz, G., and Zagier, D., *Numerical investigations related to the L -series of certain elliptic curves*, J. Indian Math. Soc. **97** (1987), 313–324.
- [Lang-LSer] Lang, S., *Sur les séries L d'une variété algébrique*, Bull. Soc. Math. France **84** (1956), 385–407.
- [Lang-Weil] Lang, S., and Weil, A., *Number of points of varieties in finite fields*, Amer. J. Math. **76** (1954), 819–827.
- [Lau-TF] Laumon, G., *Transformation de Fourier, constantes d'équations fonctionnelles et conjecture de Weil*, Inst. Hautes Études Sci. Publ. Math. **65** (1987), 131–210.
- [Mat-Mon] Matsumura, H., and Monsky, P., *On the automorphisms of hypersurfaces*, J. Math. Kyoto Univ. **3** (1964), 347–361.
- [Mehta] Mehta, M. L., *Random matrices*, Academic Press, 1991.
- [Mehta-PS] Mehta, M. L., *Power series for level spacing functions of random matrix ensembles*, Z. Phys. B: Condensed Matter **86** (1992), 285–290.
- [Mehta-Des Cloiz] Mehta M. L., and Des Cloizeaux, J., *The probabilities of several consecutive eigenvalues of a random matrix*, Indian J. Pure Appl. Math. **3** (1972), 329–351.
- [Messing] Messing, W., *The crystals associated to Barsotti-Tate groups; with applications to abelian schemes*, Lecture Notes in Math., vol. 264, Springer-Verlag, 1972.
- [Mon] Montgomery, H., *The pair correlation of zeros of the zeta function*, Analytic Number Theory (H. G. Diamond, ed.), Proc. Sympos. Pure Math., vol. 24, Amer. Math. Soc., 1973, pp. 181–193.
- [Mum-AV] Mumford, D., *Abelian varieties*, Oxford Univ. Press, 1970.
- [Mum-GIT] Mumford, D., *Geometric invariant theory*, Springer-Verlag, 1965.
- [Oda] Oda, T., *The first de Rham cohomology groups and Dieudonné modules*, Ann. Sci. École Norm. Sup. (4) **2** (1969), 63–135.

- [Odl] Odlyzko, A. M., *The 10²⁰th zero of the Riemann zeta function and 70 million of its neighbors*, ATT Bell Laboratories preprint, 1989.
- [Odl-Distr] Odlyzko, A. M., *On the distribution of spacings between zeros of the zeta function*, *Math. Comp.* **48** (1987), 273–308.
- [P-W] Pasiencier, S., and Wang, H.-C., *Commutators in a complex semi-simple Lie group*, *Proc. Amer. Math. Soc.* **13** (1962), 907–913.
- [Poon-Curves] Poonen, B., *Curves over finite fields without extra automorphisms*, preprint, November, 1996; available from <http://math.berkeley.edu/~poonen/>.
- [Poon-Hy] Poonen, B., *Hypersurfaces over finite fields without extra automorphisms*, preprint, January, 1997; available from <http://math.berkeley.edu/~poonen/>.
- [Ray] Raynaud, M., *Caractéristique d’Euler-Poincaré d’un faisceau et cohomologie des variétés abéliennes*, Séminaire Bourbaki 1964/65, W. A. Benjamin, New York, 1966, Exposé 286.
- [Ree] Ree, R., *Commutators in semi-simple algebraic groups*, *Proc. Amer. Math. Soc.* **15** (1964), 457–460.
- [Reed-Simon] Reed, M., and Simon, B., *Methods of modern mathematical physics. I: Functional analysis*, rev. ed., Academic Press, 1980.
- [Rib] Ribet, K., *Images of semi-stable Galois representations*, Olga-Taussky-Todd: In Memoriam, *Pacific J. Math.* 1997, special issue, 277–297.
- [Riesz-Sz.-Nagy] Riesz, F., and Sz.-Nagy, B., *Functional analysis*, Ungar, New York, 1955.
- [Rudin] Rudin, W., *Real and complex analysis*, 3rd ed., McGraw-Hill, New York, 1987.
- [Ru-Sar] Rudnick, Z., and Sarnak, P., *Zeros of principal L-functions and random matrix theory*, *Duke Math. J.* **81** (1996), 269–322.
- [Serre-GACC] Serre, J.-P., *Groupes algébriques et corps de classes*, Hermann, Paris, 1959.
- [Serre-Rig] Serre, J.-P., *Rigidité du foncteur de Jacobi d’échelon $n \geq 3$* , appendice d’exposé 17, Séminaire Henri Cartan 13e année, 1960/61.
- [SGA] A. Grothendieck et al., *Séminaire de Géométrie Algébrique du Bois-Marie*, SGA 1, SGA 4 Parts I, II, and III, SGA 4₂, SGA 5, SGA 7 Parts I and II, *Lecture Notes in Math.*, vols. 224, 269–270–305, 569, 589, 288–340, Springer-Verlag, 1971 to 1977.
- [Shoda] Shoda, K., *Einige Satz über Matrizen*, *Japan J. Math.* **13** (1937), 361–365.
- [Sil] Silverman, J., *The average rank of an algebraic family of elliptic curves*, *J. Reine Angew. Math.* **504** (1998) (to appear).
- [Slep-Pol] Slepian, D., and Pollak, H. O., *Prolate spheroidal wave functions, Fourier analysis and uncertainty. I*, *Bell System Tech. J.* **40** (1961), 43–64.
- [Sug] Sugiura, M., *Unitary representations and harmonic analysis—An introduction*, Kodansha, Tokyo, and Halsted Press, New York, 1975.
- [Sut] Sutor, R., *The calculation of some geometric monodromy groups*, Princeton University Ph.D. thesis, 1992.
- [Szego] Szego, S. G., *Orthogonal polynomials*, AMS Colloquium Publ., vol. 23, Amer. Math. Soc., 1939.
- [T-W] Tracy, C., and Widom, H., *Introduction to random matrices*, *Geometric and Quantum Aspects of Integrable Systems (Scheveningen 1992)*, *Lecture Notes in Physics*, vol. 424, Springer-Verlag, 1993, pp. 103–130.
- [VB] Van Buren, A. L., *A Fortran computer program for calculating the linear prolate functions*, Report 7994, Naval Research Laboratory, Washington, DC, May, 1976.
- [Weil-CA] Weil, A., *Courbes algébriques et variétés abéliennes*, Hermann, Paris, 1971.
- [Weil-NS] Weil, A., *Number of solutions of equations in finite fields*, *Bull. Amer. Math. Soc.* **55** (1949), 497–508.
- [Weyl] Weyl, H., *Classical groups*, Princeton Univ. Press, 1946.
- [W-W] Whittaker, E. T., and Watson, G. N., *A course of modern analysis*, 4th ed., reprinted, Cambridge Univ. Press, 1962.
- [Widom] Widom, H., *The asymptotics of a continuous analogue of orthogonal polynomials*, *J. Approx. Theory* **77** (1994), 51–64.
- [Yu] Yu, J.-K., *Lectures at Princeton University*, February 1995, unpublished.

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