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Global Solutions of Nonlinear Schrödinger Equations

J. Bourgain

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Global Solutions of Nonlinear Schrödinger Equations

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ABSTRACT. The aim of this book is to describe recent progress on various issues in the theory of nonlinear dispersive equations, primarily the nonlinear Schrödinger equation (NLS). In particular, the Cauchy problem for the defocusing critical NLS with radial data is discussed. New techniques and results are described on global existence of large data solutions below the energy norm. Current research in Harmonic Analysis around Strichartz' inequalities and its relevance to nonlinear PDE is presented. Also several topics in NLS theory on bounded domains are reviewed. In this respect, a partial survey is given of the theory of invariant Gibbs measures and recent developments in KAM theory for PDE's.

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