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Attractors for Equations of Mathematical Physics

Vladimir V. Chepyzhov
Mark I. Vishik
ABSTRACT. The authors study new problems related to the theory of infinite-dimensional dynamical systems that were intensively developed during the last few years. They construct the attractors and study their properties for various non-autonomous equations of mathematical physics: the 2D and 3D Navier–Stokes systems, reaction-diffusion systems, dissipative wave equations, the complex Ginzburg–Landau equation, and others. Since the attractors usually have infinite dimension, the research is focused on the Kolmogorov $\varepsilon$-entropy of attractors. Upper estimates for the $\varepsilon$-entropy of uniform attractors of non-autonomous equations in terms of $\varepsilon$-entropy of time-dependent coefficients of the equation are proved.

The authors also construct attractors for those equations of mathematical physics for which the solution of the corresponding Cauchy problem is not unique or the uniqueness is not known (for example, the 3D Navier–Stokes system). The theory of the trajectory attractors for these equations is developed, which is later used to construct global attractors for equations without uniqueness. The method of trajectory attractors is applied to the study of finite-dimensional approximations of attractors. The perturbation theory for trajectory and global attractors is developed and used in the study of the attractors of equations with terms rapidly oscillating with respect to spatial and time variables. It is shown that the attractors of these equations are contained in a thin neighbourhood of the attractor of the averaged equation.

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10 9 8 7 6 5 4 3 2 1 07 06 05 04 03 02
Dedicated to our wives, Katya and Asya
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Contents

Introduction 1

Part 1. Attractors of Autonomous Equations 15

Chapter I. Attractors of Autonomous Ordinary Differential Equations 17
  1. Semigroups and attractors 17
  2. Examples of ordinary differential equations and their attractors 21

Chapter II. Attractors of Autonomous Partial Differential Equations 27
  1. Function spaces and embedding theorems 28
  2. Operator semigroups. Basic notions 35
  3. Attractors of semigroups 37
  4. Reaction-diffusion systems 38
  5. 2D Navier–Stokes system 46
  6. Hyperbolic equation with dissipation 49

Chapter III. Dimension of Attractors 51
  1. Fractal and Hausdorff dimension 51
  2. Dimension of invariant sets 53
  3. Optimization of the bound for the fractal dimension 59
  4. Application to semigroups 62
  5. Applications to evolution equations 65
  6. Lower bounds for the dimension of attractors 73

Part 2. Attractors of Non-autonomous Equations 77

Chapter IV. Processes and Attractors 79
  1. Symbols of non-autonomous equations 80
  2. Cauchy problem and processes 82
  3. Uniform attractors 83
  4. Haraux’s example 85
  5. The reduction to a semigroup 86
  6. On uniform (w.r.t. \( \tau \in \mathbb{R} \)) attractors 92

Chapter V. Translation Compact Functions 95
  1. Almost periodic functions 95
  2. Translation compact functions in \( C(\mathbb{R}; \mathcal{M}) \) 97
  3. Translation compact functions in \( L_{p; w}^{\text{loc}}(\mathbb{R}; \mathcal{E}) \) 101
  4. Translation compact functions in \( L_{\text{cloc}}^{\text{loc}}(\mathbb{R}; \mathcal{E}) \) 104
  5. Other translation compact functions 106
Chapter VI. Attractors of Non-autonomous Partial Differential Equations 107
1. 2D Navier–Stokes system 107
2. Non-autonomous reaction-diffusion systems 114
3. Non-autonomous Ginzburg–Landau equation and others 118
4. Non-autonomous damped hyperbolic equations 119

Chapter VII. Semiprocesses and Attractors 129
1. Families of semiprocesses and their attractors 129
2. On the reduction to the semigroup 132
3. Non-autonomous equations with tr.c. on $\mathbb{R}_+$ symbols 135
4. Prolongations of semiprocesses to processes 137
5. Asymptotically almost periodic functions 140
7. Cascade systems and their attractors 146

Chapter VIII. Kernels of Processes 149
1. Properties of kernels 149
2. On the dimension of connected sets 153
3. Dimension estimates for kernel sections 155
4. Applications to non-autonomous equations 157

Chapter IX. Kolmogorov $\varepsilon$-Entropy of Attractors 163
1. Estimates of the $\varepsilon$-entropy 163
2. Fractal dimension of attractors 173
3. Functional dimension and metric order 176
4. Applications to evolution equations 177
5. $\eta$-entropy and metric order of $\Sigma$ 188
6. $\varepsilon$-entropy in the extended phase space 192

Part 3. Trajectory Attractors 197

Chapter X. Trajectory Attractors of Autonomous Ordinary Differential Equations 199
1. Preliminary propositions 200
2. Construction of the trajectory attractor 203
3. Examples of equations 205
4. Dependence on a parameter 207

Chapter XI. Attractors in Hausdorff Spaces 211
1. Some topological preliminaries 211
2. Semigroups in topological spaces and attractors 214
3. Applications to $(\mathcal{M}, \Sigma)$-attractors 218

Chapter XII. Trajectory Attractors of Autonomous Equations 219
1. Trajectory spaces of evolution equations 219
2. Existence of trajectory attractors 222
3. Trajectory and global attractors 224

Chapter XIII. Trajectory Attractors of Autonomous Partial Differential Equations 229
1. Autonomous Navier–Stokes systems 229
## CONTENTS

2. Autonomous hyperbolic equations 242
3. Hyperbolic equations depending on a parameter 251

Chapter XIV. Trajectory Attractors of Non-autonomous Equations 259
1. Non-autonomous equations, their symbols, and trajectory spaces 260
2. Existence of uniform trajectory attractors 262
3. Equations with symbols on the semiaxis 266

Chapter XV. Trajectory Attractors of Non-autonomous Partial Differential Equations 269
1. Non-autonomous Navier–Stokes systems 269
2. Trajectory attractor for 2D Navier–Stokes system 278
3. Reaction-diffusion systems 282
4. Non-autonomous hyperbolic equations 292

Chapter XVI. Approximation of Trajectory Attractors 299
1. Trajectory attractors of non-autonomous ordinary differential equations 299
2. Trajectory attractors of Galerkin systems 302
3. Convergence of trajectory attractors of Galerkin systems 303

Chapter XVII. Perturbation of Trajectory Attractors 305
1. Trajectory attractors of perturbed equations 305
2. Dependence of trajectory attractors on a small parameter 307

Chapter XVIII. Averaging of Attractors of Evolution Equations with Rapidly Oscillating Terms 311
1. Averaging of rapidly oscillating functions 311
2. Averaging of equations and systems 320
3. Perturbation with rapidly oscillating terms 341

Appendix A. Proofs of Theorems II.1.4 and II.1.5 345

Appendix B. Lattices and Coverings 349

Bibliography 353

Index 361
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Bibliography

BIBLIOGRAPHY


[38] , Trajectory attractors for reaction-diffusion systems, Topol. Methods Nonlinear Anal. 7 (1996), no. 1, 49–76.


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Index

$(E \times \Sigma, E)$-continuous family, 88
$(\Theta^loc, \Sigma)$-closed family, 262
$\varepsilon$-entropy, 51, 165
$\varepsilon$-period, 95
d-dimensional Hausdorff measure, 52
$k$-dimensional torus, 82
$m$-dimensional trace, 62

Almost periodic
function, 81, 95
   asymptotically, 133, 140
   in the Stepanov sense, 97

Attracting property, 84
Attractor, 19, 217, 218
   $(\mathcal{M}, \mathcal{T})$-attractor, 218
   global, 19, 37, 225, 239, 249
   uniform, 265
Lorenz, 65
   non-uniform, 85
   trajectory, 203, 223
   uniform, 262
   uniform, 84, 93

Average
   in $L^\infty(\Omega)$, 312
   in $L^p, w(\Omega)$, 311
   time uniform, 316

Averaging
   spatial, 311
   time, 316

Backward uniqueness property, 138
Belousov–Zhabotinsky equations, 43
Bochner–Amerio criterion, 96

Cascade system, 133
Chafee–Infante equation, 41, 330
Closure, 212

Compactness
   criterion
   in $C(\mathbb{R}; \mathcal{M})$, 98
   in $L^loc(\mathbb{R}; \mathcal{E})$, 101
   in $L^loc_p(\mathbb{R}; \mathcal{E})$, 105
   theorems, 31

Compactum, 214
Complete trajectory, 19, 38, 88, 218, 223, 263
Continuous mapping, 213

Convergent sequence, 212
   $*$-weakly, 32
   weakly, 32

Covering, 212
   density, 349
   radius, 349

Derivative in the distribution sense, 31
Differential inequality, 35

Dimension
   fractal, 52, 173
   functional, 176
   Hausdorff, 52
   local
   fractal, 175
   functional, 176
   Lyapunov, 62

Dissipative wave equation, 334
Dissipativity condition, 17
Douady–Oesterlé theorem, 55

Energy norm, 50
Equilibrium point, 20

First Urysohn theorem, 214
Fitz–Hugh–Nagumo equations, 41
Fréchet–Urysohn space, 213
Fundamental
   parallelepiped, 349
   region, 349

Gagliardo–Nirenberg inequality, 30
Galerkin
   approximation, 23, 302
   method, 231, 284

Ginzburg–Landau equation, 42, 118, 328
Grashof number, 47, 235
Gronwall’s inequality, 34
Group, 36

Hölder’s inequality, 34
Hahn–Banach theorem, 32
Haraux’s example, 85
Hausdorff
   dimension, 52
   space, 213

Hull, 81, 96, 132, 135

Hyperbolic equation
damped, 119
  dissipative, 49
  with dissipation, 49, 71, 159, 185, 292, 306
Inductive limit, 221
Instability index, 73
Interpolation inequality, 30

Kernel
  of equation, 20, 223, 263
  of process, 88, 149
  of semigroup, 38, 218
  section, 20, 38, 88, 218
Kolmogorov ε-entropy, 164

Ladyzhenskaya’s inequality, 46, 230, 235
Lattice, 349
  cube, 351
  determinant, 349
  enerating matrix, 349
  main Voronoi, 351
Lieb–Thirring inequality, 69
Lipschitz condition, 165
Lorenz
  attractor, 65
  system, 23
Lotka–Volterra system, 44
Lyapunov
  dimension, 62
  uniform exponents, 61

Metric order, 176
  local, 176
Minimality property, 84
Multiplicative properties, 83

Navier–Stokes system, 269
  2D, 46, 68, 74, 107, 157, 177, 239, 278, 323
  3D, 229, 305, 320
Nikol’skiǐ space, 279

Periodic orbit, 20
Point
  adherent, 211
  limit, 212
Process, 82, 83
  bounded, 83
  family of processes, 84
  periodic, 87
Quasidifferential, 53
Quasiperiodic
  function, 82, 96
  solution, 20
  symbol, 88
Reaction–diffusion
  equation, 38
  system, 66, 75, 114, 158, 181, 282, 325

Second axiom of countability, 212
Second Uryson theorem, 214
Semigroup, 18, 36, 214
  (E, E)-bounded, 37
  (E, E)-continuous, 37
  asymptotically compact, 37
  compact, 37
  identity, 36
Semiprocess, 129
Set
  (M, T)-attracting, 218
  ω-limit, 19, 38, 130, 215
  absorbing, 18, 37, 83
  attracting, 37, 83, 223
  countably precompact, 214
  local unstable, 73
  precompact, 214
  relatively dense, 95
  uniformly absorbing, 84
  attracting, 84, 92, 262
Sets
  closed, 211
  open, 211
Sine–Gordon equation, 49
Sobolev embedding theorem, 29
Space
  compact, 214
  countably compact, 214
  Fréchet-Uryson, 213
  Hausdorff, 213
  metrizable, 214
  normal, 213
  separable, 212
  topological, 211
Symbol
  of equation, 79, 80
  of process, 84
  space, 80, 81, 84
Topology base, 212
Trajectory, 220, 261
  attractor, 203, 223
  uniform, 262
  space, 200, 219, 260
  united, 261
Translation
  bounded function, 105
  compact function, 81, 105, 135
  group, 260
  identity, 83, 86
  semigroup, 200
Uniformly quasidifferentiable
  map, 53
  sequence, 153
Unstable trajectory, 20
Volume contracting condition, 165
Voronoi region, 349
Weak solution, 230, 242, 283
Young's inequality, 34