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Structure of the Standard Modules for the Affine Lie Algebra $A_1^{(1)}$

James Lepowsky and Mirko Primc

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PREFACE

The Lie algebra $\mathfrak{sl}(2)$, also denoted A_1, has often served as a source of new ideas in the representation theory of finite-dimensional Lie algebras. On the one hand, its structure is typical of that of its more complicated cousins, but on the other hand, its small size permits computations that might otherwise be too difficult.

The affine Lie algebra $\mathfrak{sl}(2)\hat{}$, also denoted $A_1^{(1)}$, has recently been playing an analogous role in the representation theory of infinite-dimensional Lie algebras. In particular, it was discovered a few years ago that $\mathfrak{sl}(2)\hat{}$ and then a general class of affine Lie algebras could be constructed using operators closely related to the "vertex operators" in the physicists' string model. The calculus of vertex operators has since led to many exciting developments.

In the present paper, this calculus is developed to solve the problem of constructing all the "standard" $\mathfrak{sl}(2)\hat{}$-modules in the "homogeneous" realization. We have tried to make the exposition detailed and concrete enough to be accessible and illuminating to even a relative newcomer to the theory.

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