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1. TIGHT LATTICES

In this chapter we give the slightly technical definition of the class of "tight" lattices (Definition 1.6). Each lattice of this class, when isomorphic to an interval in the congruence lattice of a finite algebra, produces an algebraic phenomenon we call "tameness".

DEFINITION 1.1. Let $\mathbf{L} = \langle L, \vee, \wedge \rangle$ be any lattice.

(1) By a **meet endomorphism** of **L** we mean a function $\mu : L \to L$ satisfying $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$ for all elements x and y in **L**.

(2) By a **join endomorphism** of **L** we mean a meet endomorphism of the dual lattice $\mathbf{L}^{\partial} = \langle L, \wedge, \vee \rangle$.

(3) A function $\mu : L \to L$ is **increasing** iff $\mu(x) \geq x$ for all x in L; and μ is **strictly increasing** iff $\mu(x) > x$ for all x in L except the largest element (if **L** has a largest element). The concepts of **decreasing** and of **strictly decreasing** function from **L** to **L** are defined in an analogous fashion.

(4) By a **polarity** of **L** we mean a pair $\langle \sigma, \mu \rangle$ such that σ is a decreasing join endomorphism of **L**, and μ is an increasing meet endomorphism of **L**, and $\sigma\mu(x) \leq x \leq \mu\sigma(x)$ for all x in L.

(5) By a **tolerance** of **L**, we mean a reflexive and symmetric subalgebra of $\mathbf{L}^2$, i.e., a binary relation $\rho \subseteq L^2$ such that for all $x, y, u, v \in L$ we have : (i) $\langle x, x \rangle \in \rho$; (ii) $\langle x, y \rangle \in \rho$ iff $\langle y, x \rangle \in \rho$; (iii) if $\langle x, y \rangle$, $\langle u, v \rangle \in \rho$ then $\langle x \vee u, y \vee v \rangle \in \rho$ and $\langle x \wedge u, y \wedge v \rangle \in \rho$.

There is an extensive literature on tolerances of lattices. Our use of the concept will be restricted to finite lattices, for which the basic facts we need can be easily proved. In finite lattices, there are one-one correspondences: tolerances $\leftrightarrow$ polarities $\leftrightarrow$ increasing meet endomorphisms $\leftrightarrow$ decreasing join endomorphisms.

LEMMA 1.2. *Let* **L** *be a finite lattice.*

(1) *A pair $\langle f, g \rangle$ of mappings from L into L is a polarity iff f is decreasing or g is increasing, and for all $x, y \in L$ we have $f(x) \leq y$ iff $x \leq g(y)$.*

(2) *The relation $\{\langle \sigma, \mu \rangle : \langle \sigma, \mu \rangle$ is a polarity $\}$ is a one-to-one mapping of the set of all decreasing join endomorphism of L onto the set of all increasing meet endomorphisms of* **L**.

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(3) *If ρ is any tolerance of* $\mathbf{L}$, *then the formulas* $\sigma(x) = \bigwedge\{y : \langle x, y\rangle \in \rho\}$ *and* $\mu(x) = \bigvee\{y : \langle x, y\rangle \in \rho\}$ *define a polarity* $\langle\sigma, \mu\rangle$ *such that* $\rho = \{\langle x, y\rangle : \sigma(x \vee y) \leq x \wedge y\}$.

(4) *If* $\langle\sigma, \mu\rangle$ *is any polarity of* $\mathbf{L}$, *then there is a unique tolerance* ρ *such that* σ, μ, *and* ρ *are related as in* (3).

PROOF. We begin with (1). Suppose that $\langle f, g\rangle$ is a polarity. Then f and g are order preserving and $fg(x) \leq x \leq gf(x)$ for all x. Thus if $f(x) \leq y$, then $gf(x) \leq g(y)$, i.e., $x \leq gf(x) \leq g(y)$. That $x \leq g(y)$ implies $f(x) \leq y$, is analogously proved. Now suppose only that f is decreasing or g is increasing, and that $f(x) \leq y$ iff $x \leq g(y)$ holds for all x and y. From $f(x) \leq f(x)$, it follows that $x \leq gf(x)$; and $fg(x) \leq x$ follows from $g(x) \leq g(x)$. We have $f(x) \leq x$ for all x iff $x \leq g(x)$ for all x. Thus, in fact, f is decreasing and g is increasing. Both functions are order preserving; for example, if $x \leq y$ then $x \leq gf(y)$, implying $f(x) \leq f(y)$. Finally, let us show that f is a join endomorphism. (The proof that g is a meet endomorphism is entirely analogous to the argument we now give.) We choose any x and y in L, and notice that $g(f(x) \vee f(y)) \geq gf(x) \vee gf(y)$ (since g is order-preserving), and $gf(x) \vee gf(y) \geq x \vee y$. Thus $g(f(x) \vee f(y)) \geq x \vee y$, implying that $f(x \vee y) \leq f(x) \vee f(y)$. On the other hand, $f(x \vee y) \geq f(x) \vee f(y)$ since f is order-preserving. So we have $f(x \vee y) = f(x) \vee f(y)$.

Statement (2) breaks down into two assertions: that polarity is a one-to-one correspondence, and that every increasing meet endomorphism is one half of a polarity (and dually, every decreasing join endomorphism is one half of a polarity). If $\langle\sigma, \mu\rangle$ is to be a polarity, then each of σ and μ determine the other; by (1), for example, $\sigma(x)$ can be nothing but the least element y satisfying $\mu(y) \geq x$. Now suppose that μ is any increasing meet endomorphism. Define $\sigma(x) = \bigwedge\{y : \mu(y) \geq x\}$. Since the meet is a meet of finitely many elements, and μ is a meet endomorphism, we have $x \leq \mu\sigma(x)$ for all x. So if $\sigma(x) \leq y$ then $x \leq \mu\sigma(x) \leq \mu(y)$. If $x \leq \mu(y)$, then $\sigma(x) \leq y$ by the definition. It now follows from (1) that $\langle\sigma, \mu\rangle$ is a polarity. If we are given any decreasing join endomorphism σ, then we define $\mu(x) = \bigvee\{y : \sigma(y) \leq x\}$ and, proceeding as above, prove that $\langle\sigma, \mu\rangle$ is a polarity.

Let ρ be any tolerance of $\mathbf{L}$, and define σ and μ as in statement (3). Since ρ is reflexive, σ is decreasing and μ is increasing. Obviously, for all x we have $\langle x, \sigma(x)\rangle, \langle x, \mu(x)\rangle \in \rho$; and $\sigma(x)$ is the least element y with $\langle x, y\rangle \in \rho$, while $\mu(x)$ is the largest such element. Now if $\sigma(x) \leq y$, then from $\langle x, \sigma(x)\rangle \in \rho$ and $\langle y, y\rangle \in \rho$ we obtain $\langle x \vee y, \sigma(x) \vee y\rangle = \langle x \vee y, y\rangle \in \rho$. Thus $\langle y, x \vee y\rangle \in \rho$, and $\mu(y) \geq x \vee y \geq x$. Analogously, $\mu(y) \geq x$ implies $\sigma(x) \leq y$. By (1), it follows that $\langle\sigma, \mu\rangle$ is a polarity.

Let us show that $\langle\sigma, \mu\rangle$ determines ρ in the manner asserted. Let x and y be any elements such that $\sigma(x \vee y) \leq x \wedge y$. We have $\langle(x \vee y) \vee y, \sigma(x \vee y) \vee y\rangle \in \rho$, i.e., $\langle x \vee y, y\rangle \in \rho$; and similarly, $\langle x \vee y, x\rangle \in \rho$, implying that $\langle x, x \vee y\rangle \in \rho$. Thus $\langle x, y\rangle = \langle(x \vee y) \wedge x, y \wedge (x \vee y)\rangle$ is in ρ. Conversely, if $\langle x, y\rangle \in \rho$, then $\langle x \vee y, x\rangle$ and

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$\langle x \vee y, y\rangle$ are in ρ, and taking meets, we find that $\langle x \vee y, x \wedge y\rangle \in \rho$. Thus, it follows from the definition that $\sigma(x \vee y) \leq x \wedge y$. This concludes the proof of (3).

Statement (4) has a straightforward proof. We leave it to the reader. □

Any congruence relation of a lattice is a tolerance. If ρ is a tolerance of $\mathbf{L}$, then the transitive closure of ρ is a congruence relation of $\mathbf{L}$. We call a tolerance ρ **connected** iff its transitive closure is all of L^2.

LEMMA 1.3. *Let ρ be a tolerance of a finite lattice $\mathbf{L}$, and let $\langle \sigma, \mu\rangle$ be the associated polarity. The following are equivalent.*

(1) *ρ is connected.*
(2) *There exists a sequence $0 = x_0 < x_1 < \cdots < x_n = 1$ of elements of $\mathbf{L}$ (for some $n \geq 0$) with $\langle x_i, x_{i+1}\rangle \in \rho$ for all $i < n$.*
(3) *σ is strictly decreasing.*
(4) *μ is strictly increasing.*

PROOF. Suppose that ρ is connected. This implies that there exists a sequence $0 = y_0, \ldots, y_n = 1$ with $\langle y_i, y_{i+1}\rangle \in \rho$ for $i < n$. Define $x_i = \bigvee\{y_j : j \leq i\}$ when $i \leq n$. Then for $i < n$, we have $\langle x_i, x_{i+1}\rangle = \langle x_i \vee y_i, x_i \vee y_{i+1}\rangle$, and so $\langle x_i, x_{i+1}\rangle \in \rho$; and obviously $x_i \leq x_{i+1}$, and $x_0 = 0$, $x_n = 1$. By removing any repeated terms from this sequence, we obtain a strictly increasing sequence. Thus (1) implies (2).

To prove that (2) implies (3), let the sequence $0 = x_0 < \cdots < x_n = 1$ be given as in (2), and let $x > 0$ in L. There is an $i < n$ such that $x_{i+1} \geq x$ and $x_i \not\geq x$. For this i, we have $\langle x, x \wedge x_i\rangle = \langle x \wedge x_{i+1}, x \wedge x_i\rangle \in \rho$, and consequently, $\sigma(x) \leq x \wedge x_i < x$. Thus σ is strictly decreasing.

Now suppose that σ is strictly decreasing. Let x be any element of L such that $x < 1$. Choose z to be a minimal member of $\{y \in L : y \not\leq x\}$. We have $z \neq 0$, consequently $\sigma(z) < z$, implying $\sigma(z) \leq x$. This last inclusion is equivalent to $z \leq \mu(x)$. Since $z \not\leq x$, it follows that $\mu(x) > x$. Thus μ is strictly increasing.

The proof that (4) implies (1) is a simple matter of considering the sequence $0 < \mu(0) < \mu\mu(0) < \cdots$. □

For two elements x and y of a lattice (or of a partially ordered set) recall that y **covers** x (in symbols $x \prec y$) iff $x < y$ and whenever $x \leq z \leq y$, either $z = x$ or $z = y$. If L has 0 (a smallest element) then an **atom** of L is simply any element u such that $0 \prec u$. A **dual atom of** L is an element u such that $u \prec 1$.

LEMMA 1.4. *Let $\mathbf{L}$ be a finite lattice.*

(1) *The subalgebra of $\mathbf{L}^2$ generated by $\{\langle x, x\rangle : x \in L\} \cup \{\langle x, y\rangle : x \prec y$ or $y \prec x\}$ is a connected tolerance, and it is the smallest connected tolerance of $\mathbf{L}$.*
(2) *A meet endomorphism μ (or join endomorphism σ) of $\mathbf{L}$ is strictly increasing (or strictly decreasing) iff $\mu(x) \geq y$ (or $\sigma(y) \leq x$) whenever $x \prec y$ in $\mathbf{L}$.*

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PROOF. The relation defined in (1) is certainly a connected tolerance. Let ρ be any connected tolerance of L, and let $0 < x_1 < \cdots < x_n = 1$ be a sequence with $\langle x_i, x_{i+1}\rangle \in \rho$ for all i. Let $a \prec b$ be any covering in L. There exists an i such that $b \wedge x_i \leq a$ and $b \wedge x_{i+1} \not\leq a$. For this i, we have $\langle a, b\rangle = \langle a \vee (b \wedge x_i),\ a \vee (b \wedge x_{i+1})\rangle$, which implies that $\langle a, b\rangle \in \rho$. Thus ρ contains the tolerance generated by the covering relation. This proves (1); and (2) follows easily from (1), by Lemmas 1.2 and 1.3. □

DEFINITION 1.5. Let $\mathbf{L}$ be any lattice with 0 and 1. A homomorphism $f : \mathbf{L} \to \mathbf{L}'$ is called 0, 1-**separating** iff $f^{-1}\{f(0)\} = \{0\}$ (f separates 0) and $f^{-1}\{f(1)\} = \{1\}$ (f separates 1). To denote that f is a homomorphism with the property just defined, we write $f : \mathbf{L} \overset{0,1-\text{sep}}{\longrightarrow} \mathbf{L}'$. We say that $\mathbf{L}$ is 0, 1-**simple** iff $|L| > 1$ and every non-constant homomorphism $f : \mathbf{L} \to \mathbf{L}'$ ($\mathbf{L}'$ any lattice) is 0, 1–separating.

DEFINITION 1.6. A lattice $\mathbf{L}$ will be called **tight** iff $\mathbf{L}$ is finite, $|L| > 1$, and if ρ is any tolerance of $\mathbf{L}$ such that ρ contains $\langle 0, a\rangle$ for some $a > 0$ in $\mathbf{L}$, or ρ contains $\langle b, 1\rangle$ for some $b < 1$ in $\mathbf{L}$, then $\rho = L^2$.

The whole purpose of this chapter is to define tight lattices and to collect the facts about them that will be needed later on.

LEMMA 1.7. *A finite lattice* $\mathbf{L}$ *is tight iff* $\mathbf{L}$ *is* 0, 1*-simple and every strictly increasing meet endomorphism of* $\mathbf{L}$ *is constant (i.e.,* L^2 *is the only connected tolerance of* $\mathbf{L}$*).*

PROOF. Assume that $\mathbf{L}$ is tight. It follows from Lemma 1.4 and Definition 1.6 that $\mathbf{L}$ has only the trivial connected tolerance. From Lemmas 1.2 and 1.3, every strictly increasing meet endomorphism μ of $\mathbf{L}$ satisfies $\mu(0) = 1$ (i.e., μ is constant). Let $f : \mathbf{L} \to \mathbf{L}'$ be a non-constant homomorphism, and let $\theta = \ker f = \{\langle x, y\rangle : f(x) = f(y)\}$. Thus θ is a tolerance of $\mathbf{L}$, in fact a congruence. Since $\theta \neq L^2$, it follows from the definition of tight lattice, that $\langle 0, x\rangle \notin \theta$ for any $x > 0$ in $\mathbf{L}$. This means that f is 0–separating. Similarly, we can prove that it is 1–separating. So we conclude that $\mathbf{L}$ is 0, 1-simple.

Now let us assume that $|L| > 1$ and $\mathbf{L}$ is not tight. Let ρ be a tolerance of $\mathbf{L}$ such that $\rho \neq L^2$, and say, $\langle b, 1\rangle \in \rho$ for some $b < 1$. If ρ is connected, then we have a non-trivial connected tolerance of $\mathbf{L}$. If ρ is not connected, and θ is the transitive closure of ρ, then θ is a congruence, $\theta \neq L^2$, and $\langle b, 1\rangle \in \theta$. The homomorphism $\mathbf{L} \twoheadrightarrow \mathbf{L}/\theta$ is not constant and not 1–separating. Thus $\mathbf{L}$ fails to be 0, 1-simple. □

LEMMA 1.8. *For any lattice* $\mathbf{L}$ *with* 0 *and* 1*, such that* $|L| > 1$*, the following are equivalent.*

(1) $\mathbf{L}$ *is* 0, 1*-simple.*
(2) $\mathbf{L}$ *has a largest congruence* $\theta \neq L^2$*, and this congruence satisfies* $1/\theta = \{1\}$, $0/\theta = \{0\}$.

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PROOF. Suppose that L is 0,1-simple. Define $\theta = \bigvee\{\psi \in \operatorname{Con}\mathbf{L} : \psi \neq L^2\}$. We claim that $\theta \neq L^2$, in fact $1/\theta = \{1\}$ and $0/\theta = \{0\}$. To see it, suppose, for example, that $\langle x, 1\rangle \in \theta$. Now θ, the complete join in the lattice $\operatorname{Con}\mathbf{L}$ of all proper congruences of $\mathbf{L}$, is just the transitive closure of the relation $\rho = \bigcup\{\psi : \psi \neq L^2\}$. Therefore there exists a sequence $x = x_0, x_1, \ldots, x_n = 1$ such that for all $i < n$, $\langle x_i, x_{i+1}\rangle \in \psi_i$ for some congruence ψ_i of $\mathbf{L}$ with $\psi_i \neq L^2$. Since $\mathbf{L}$ is 0,1-simple, the map $\mathbf{L} \twoheadrightarrow \mathbf{L}/\psi_i$ is 1–separating, equivalently, $1/\psi_i = \{1\}$. Therefore $x_{n-1} = 1$, and then $x_{n-2} = 1$, and so on, leading to $x = 1$. We can conclude that $1/\theta = \{1\}$. Thus $\theta \neq L^2$, and it follows that θ is the largest congruence of L which is $\neq L^2$. That $0/\theta = \{0\}$ is proved as above. Thus (1) implies (2).

Suppose, now, that (2) holds. Let $f : \mathbf{L} \to \mathbf{L}'$ be any non-constant homomorphism. Then $\ker f \subseteq \theta$, implying that f is 0,1–separating. Thus (1) holds. □

Exercises 1.9

(1) Show that, among the lattices pictured below, $\mathbf{M}_n$ ($n \geq 3$) and $\mathbf{C}_2$ are tight, while the others are not.

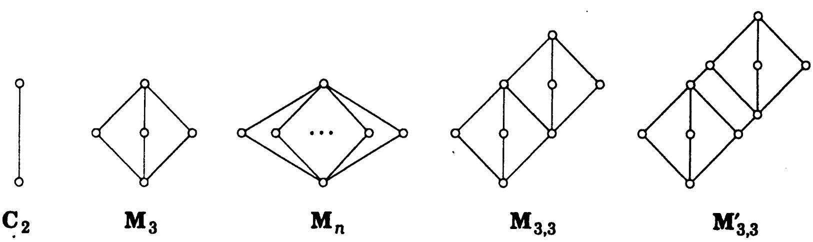

Figure 1

(2) Show that if $f : \mathbf{L} \overset{0,1-\mathrm{sep}}{\longrightarrow} \mathbf{L}'$ is surjective and if $\mathbf{L}$ is 0,1-simple, then so is $\mathbf{L}'$. Use the lattices $\mathbf{M}'_{3,3}$ and $\mathbf{M}_{3,3}$ to show that $\mathbf{L}$ may fail to be 0,1-simple even if $\mathbf{L}'$ is, in this situation.

LEMMA 1.10.

(1) *Let $f : \mathbf{L} \overset{0,1-\mathrm{sep}}{\longrightarrow} \mathbf{L}'$ be surjective, where $\mathbf{L}$ and $\mathbf{L}'$ are finite. Then $\mathbf{L}$ is tight iff $\mathbf{L}'$ is tight.*

(2) *A finite lattice $\mathbf{L}$ is tight iff there exists a simple tight lattice $\mathbf{L}'$ and a surjective 0,1–separating homomorphism of $\mathbf{L}$ onto $\mathbf{L}'$. When they exist, $\mathbf{L}'$ is determined up to isomorphism; and the kernel of f is the unique dual atom of* $\operatorname{Con}\mathbf{L}$.

PROOF. Suppose that $f : \mathbf{L} \overset{0,1-\mathrm{sep}}{\longrightarrow} \mathbf{L}'$ is surjective and $\mathbf{L}$ is finite. Assume that $\mathbf{L}'$ is tight, and let ρ be a tolerance of $\mathbf{L}$ such that, say, $\langle 0, a\rangle \in \rho$ for some element $a > 0$ of $\mathbf{L}$. We define $\rho' = f(\rho) = \{\langle f(x), f(y)\rangle : \langle x, y\rangle \in \rho\}$, and verify that ρ' is

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a tolerance of $\mathbf{L}'$ and $\langle 0', b'\rangle \in \rho'$, where $0' < b' = f(a)$ and $0'$ is the zero element of $\mathbf{L}'$. Therefore $\rho' = (L')^2$; and from this we conclude that for some $\langle u, v\rangle \in \rho$, $\langle f(u), f(v)\rangle = \langle 0', 1'\rangle$. Since f is $0,1$–separating, we must have $\langle u, v\rangle = \langle 0, 1\rangle$. The tolerance ρ, containing $\langle 0, 1\rangle$, can be nothing other than L^2. Thus $\mathbf{L}$ is tight if $\mathbf{L}'$ is tight. Now assume that $\mathbf{L}$ is tight. Let ρ' be a tolerance of $\mathbf{L}'$ containing, say, $\langle 0', c'\rangle$ for some $c' > 0'$. We define $\rho = f^{-1}(\rho') = \{\langle x, y\rangle \in L^2 : \langle f(x), f(y)\rangle \in \rho'\}$, and verify that ρ is a tolerance and $\langle 0, c\rangle \in \rho$ for some $c > 0$. Therefore $\rho = L^2$, and consequently $\rho' = (L')^2$. Thus $\mathbf{L}'$ is tight. This ends the proof of (1).

To prove (2), suppose first that $\mathbf{L}$ is tight. Let θ be the largest proper congruence of $\mathbf{L}$ (which exists by Lemma 1.8). The natural homomorphism $f : \mathbf{L} \twoheadrightarrow \mathbf{L}/\theta$ is $0,1$–separating (since $\mathbf{L}$ is $0,1$-simple), and $\mathbf{L}/\theta$ is a simple lattice, which is tight by statement (1). Now assume, conversely that $g : \mathbf{L} \twoheadrightarrow \mathbf{L}'$ is $0,1$–separating and $\mathbf{L}'$ is simple and tight. That $\mathbf{L}$ is tight follows from (1). The kernel of g can be nothing other than the unique dual atom of $\operatorname{Con}\mathbf{L}$. (Since $\mathbf{L}'$ is simple, $\ker g$ is a dual atom.) Thus $\mathbf{L}' \cong \mathbf{L}/\ker g$ is determined up to isomorphism by $\mathbf{L}$. □

The class of tight lattices is quite diverse, as can be seen from these examples.

Example 1.11. A finite simple (or $0,1$-simple) lattice, satisfying any one of the following conditions, is tight. (i): $\mathbf{L}$ is complemented (i.e., for all $x \in L$ there exists $x' \in L$ such that $x \vee x' = 1$ and $x \wedge x' = 0$). (ii): The atoms of $\mathbf{L}$ join to 1. (iii): The dual atoms of $\mathbf{L}$ meet to 0. In fact, condition (i) implies (ii) and (iii); and each of (ii) and (iii) implies that $\mathbf{L}$ cannot have a non-constant, strictly increasing, meet endomorphism (by Lemma 1.4 (2)). A finite $0,1$-simple lattice satisfying one of these conditions is tight (by Lemma 1.7).

Example 1.12. For each integer $n \geq 2$, the lattice $\mathbf{\Pi}_n$ of all equivalence relations on an n-element set is tight. These lattices are simple and complemented.

Example 1.13. For each prime p and integers $k, n \geq 1$, the lattice $\mathbf{S}(p^k, n)$ of all subspaces of an n-dimensional vector space over a finite field of p^k elements, is tight. This lattice is simple and complemented. It is isomorphic to the lattice of congruence relations of the vector space.

Any finite lattice that admits a $0,1$–separating homomorphism onto a lattice $\mathbf{\Pi}_n$ or $\mathbf{S}(p^k, n)$ is tight (by Lemma 1.10). According to Exercise 1.14 (1), such lattices are $0,1$-simple and complemented. Some tight lattices are pictured on the next page.

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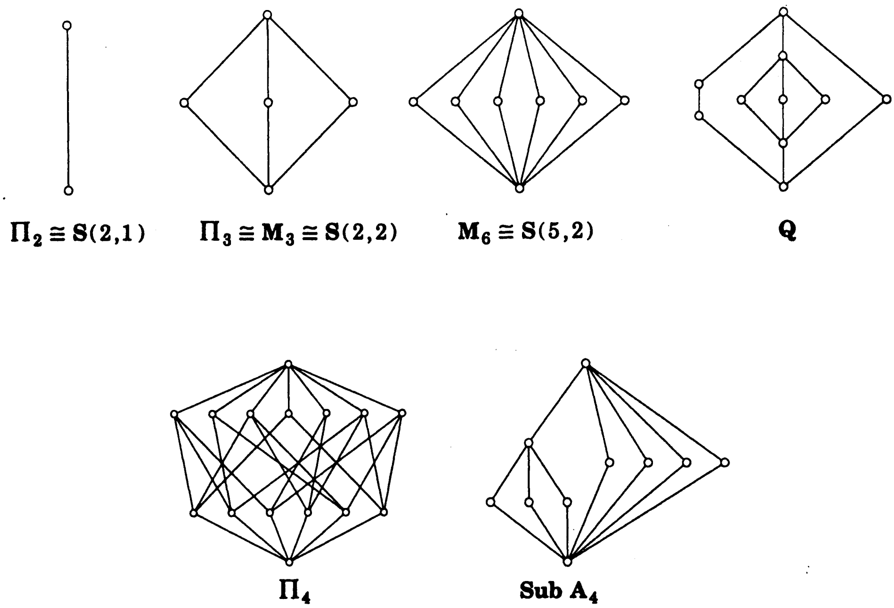

Figure 2

Sub $\mathbf{A}_4$ is the lattice of subgroups of the twelve-element alternating group. The lattice $\mathbf{Q}$ admits an obvious $0,1$–separating homomorphism onto $\mathbf{M}_3$.

The first substantial result proved as an application of tame congruence theory was that **Sub** $\mathbf{A}_4$ (and many other tight lattices, in particular $\mathbf{M}_n$ if $n \geq 3$) cannot be isomorphic to the congruence lattice of any finite algebra with just one basic operation, such as a semigroup. (This result will not be proved in this book. The proof can be found in [22].)

We shall prove in Chapter 5 (Theorem 5.7 (4)) that, except in quite unusual situations, if an interval sublattice $\mathbf{L}$ of the congruence lattice of a finite algebra is tight, then $\mathbf{L}$ admits a $0,1$–separating homomorphism onto the lattice of subspaces of a finite vector space. In the next exercises, all lattices mentioned are assumed to be finite.

Exercises 1.14

(1) Let $f : \mathbf{L} \overset{0,1-\text{sep}}{\longrightarrow} \mathbf{L}'$ be surjective. For each of conditions (i), (ii), (iii) of Example 1.11, show that $\mathbf{L}$ satisfies the condition iff $\mathbf{L}'$ satisfies it.

(2) The full partition lattice $\mathbf{\Pi}_n$ of Example 1.12 is simple and complemented, if $n \geq 2$.

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(3) $\mathbf{Con\,V}$ is simple and complemented whenever $\mathbf{V}$ is a finite vector space of more than one element. (It will be easier to work with the isomorphic lattices $\mathbf{S}(p^k, n)$ of Example 1.13.)

(4) Let $\mathbf{L}$ be modular. Show that the formula $\mu(x) = \bigvee\{y : x \prec y\}$ defines a meet endomorphism of $\mathbf{L}$. [By Lemma 1.4, the associated tolerance is the smallest connected tolerance of $\mathbf{L}$. This exercise is harder than most. The key is to prove that $x \leq \mu(y)$ iff $y \geq \bigwedge\{z : z \prec x\} = \sigma(x)$.]

(5) If $\mathbf{L}$ is modular then $\mathbf{L}$ is tight iff $\mathbf{L}$ is simple and complemented.

(6) $\mathbf{L}$ is tight iff the dual lattice, $\mathbf{L}^\partial$, is tight.

The next five exercises were contributed by Brian Davey and Emil Kiss. A sublattice of $\mathbf{L}^2$ containing the diagonal $\Delta = \{\langle x, x\rangle : x \in \mathbf{L}\}$ will be called a *diagonal sublattice.* $\mathbf{L}$ is called *order polynomially complete* iff every monotone mapping $L^n \to L$ (for any n) is a polynomial operation of $\mathbf{L}$. The diagonal sublattice generated by a pair $\langle a, b\rangle \in L^2$ is denoted $L(a, b)$.

(7) $\mathbf{L}$ is tight iff for every $a \neq 0$ and $b \neq 1$, $\langle 0, 1\rangle \in L(0, a) \cap L(b, 1)$.

(8) $\mathbf{L}$ is tight iff every diagonal sublattice of $\mathbf{L}^2$ is of the form K, $\leq \circ K$, $\geq \circ K$, or L^2, where K is a $0,1$–separating diagonal sublattice (i.e., $\langle a, 0\rangle$ or $\langle 0, a\rangle$ in K implies $a = 0$, and $\langle b, 1\rangle$ or $\langle 1, b\rangle$ in K implies $b = 1$).

(9) If $\mathbf{L}$ is simple then Δ is the only $0,1$–separating diagonal sublattice of $\mathbf{L}^2$.

(10) $\mathbf{L}$ is order polynomially complete iff Δ, $\leq$, $\geq$, L^2 are the only diagonal sublattices of $\mathbf{L}^2$.

(11) $\mathbf{L}$ is order polynomially complete iff $\mathbf{L}$ is tight and simple.

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