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# On Partial Cartesian Closed Categories.

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## Introduction.

Recently the study of partial categorical notions has been subject by several people ( [Cu.Ob.], [Di.He.], [Ro.]). The main interest in this paper is to clarify the relationship between Partial Cartesian Categories, Partial Cartesian Closed Categories, and Cartesian, Cartesian Closed Categories. We will follow the spirit of [Cu.Ob.] and we will try to answer some natural questions which arise after reading that paper.

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## 1. Categories with partial morphisms.

1.1 Definition [Cu.Ob.]. A category with partial morphisms or $p\underline{C}$ is a category $\underline{C}$ such that for every homset $hom_{\underline{C}}(A, B)$ we have a partial order structure and has a distinguished subset of maximal arrows called total; composition is monotonic.

Total arrows satisfy the following: Identity arrows are total, total arrows are closed under composition.

1.2 Examples.

Every pointed category [Di.He.] is a $p\underline{C}$-category. Every category with finite limits is a $p\underline{C}$-category.

For every $p\underline{C}$- category $\underline{C}$, we can consider the subcategory $\underline{C}_T$ of $\underline{C}$ whose objects are the same as $\underline{C}$ and whose morphisms are the total arrows. We are interested in sudying the relationship between the notions of partial product, partial exponentiation, for $\underline{C}$ and the corresponding notions (products, exponentiation) for $\underline{C}_T$.

1.3 Definition [Cu.Ob.]. A $p\underline{C}$ category $\underline{C}$ has liftings if for every $\underline{C}$-object $A$ there is an object $\tilde{A}$ called the lifting of $A$, such that for every object $B$ the is a bijection $\sigma_{B,A} : hom_{\underline{C}}(B, A) \to hom_{\underline{C}_T}(B, \tilde{A})$ such that for all $f : C \to B$ and $g : B \to A$ in $\underline{C}$ we have:

$$\sigma_{B,A}(g)f \leq \sigma_{C,A}(gf).$$

Perhaps, liftings are motivated by the following example. If we start with any topos $\underline{E}$ and consider the category of partial morphisms then $\tilde{A}$ is the partial map classifier of $A$.

Notice that when a $p\underline{C}$-category $\underline{C}$ has liftings then for every $\underline{C}$-object $A$ we have an inclusion

$$\eta_A : A \to \tilde{A}$$

where $\eta_A = \sigma_{A,A}(1_A)$ ; in fact, this arrow is a $\underline{C}$-section via $ex_A = \sigma^{-1}_{\tilde{A},A}(1_A)$; Moreover,$\eta_A$ is always a total morphism but $ex_A$ is ot necessaraly a total morphism.We will say that $A$ is a complete object when in fact there is a total map

$$p : \tilde{A} \to A$$

such that $p\eta_A = 1_A$.

Curien and Obtulowicz study in detail complete objects; for instance,

$$\tilde{\ }\underline{C} \longrightarrow \underline{C}_T$$

is a functor and it is right adjoint to the inclusion functor $i : \underline{C}_T \longrightarrow \underline{C}$. In particular, the following proposition will be very useful.

1.4 Proposition [Cu.Ob.] An object $A$ is complete if and only if, for every object $B$ and every morphism $f : B \to A$, there is a total morphism $g : B \to A$ such that $f \le g$. In particular, $\tilde{A}$ is complete for every object $A$.

## 2.Partial Cartesian and Cartesian Closed Categories.

We introduce now the notion of partial cartesian category.

2.1 Definition [Cu.Ob.] A $p\underline{C}$-category $\underline{C}$ is called partially cartesian or $p\underline{CC}$ when it has the following structure:

I.-A partial terminal object 1 and for every object $A$, an arrow $!_A : A \to 1$ satisfying

i. $!_A$ is maximum in $hom(A,1)$ for every object $A$.

ii. The total morphisms are exactly those $f : A \to B$ such that $!_B f = !_A$.

iii. If $f, f', g : A \to B$ are such that $f, f' \le g$ and $!_B f \le !_B f'$ then $f \le f'$.

iv. If $h, h' : A \to B$ are such that $h \le h'$ then for any $g : B \to 1$ we have $gh = (gh') \cap (gh)$.

II.- For every object $A, B$ a partial product given by an object $A \times B$, partial projections $\pi_{A,B} : A \times B \to A$, $\pi'_{A,B} : A \times B \to B$, and such that for every pair of morphisms $f : C \to A, g : C \to B$ there is a morphism $\langle f, g\rangle : C \to A \times B$, satisfying:

v. $\pi_{A,B}, \pi'_{A,B}$ are total morphisms.

vi. $\langle -, -\rangle$ is monotonic in both arguments.

vii. For any $h : C \to A \times B$ , $h = \langle \pi_{A,B}h, \pi'_{A,B}h\rangle$

viii. $\pi_{A,B}\langle f, g,\rangle \le f$ and $\pi'_{A,B}\langle f, g\rangle \le g$.

ix. For all $k : D \to C, \langle f, g\rangle k = \langle fk, gk\rangle$.

x. $1_{A\times B}\langle fg,\rangle = (!_B f) \cap (!_B g)$.

2.2 Example.

Every category with finite limits is a partial cartesian category.

An interesting consequence of this definition, as is pointed out in [Cu.Ob.], is that $\underline{C}_T$ is actually a Cartesian Category.

We are now ready to introduce the notion of partial cartesian closed category.

2.3 Definition [Cu.Ob.]. A partial cartesian category $\underline{C}$ is partially cartesian closed or $p\underline{CCC}$ if for any objects $A, B$ there is an object $A^B$ such that for every object $C$ there is a bijection

$$\lambda : hom_{\underline{C}}(C \times B, A) \to hom_{\underline{C_T}}(C, A^B)$$

satisfying the following condition:

For all $f : D \to C, g : C \times B \to A$

$$\lambda(g)f \leq \lambda(g(f \times 1_B))$$

where $f \times 1_B \equiv \langle f\pi_{D,B}, \pi'_{D,B}\rangle$.

2.4 Example. If $\underline{E}$ is any topos and we consider the category of partial morphisms $\underline{E}_p$ then $A^B$ is nothing but $(\tilde{A})^B$. This suggests that every $p\underline{CCC}$-category has liftings.

2.5 Proposition [Cu.Ob.] Every $p\underline{CCC}$-category $\underline{C}$ has liftings; namely, for every $\underline{C}$-object $A$, we have $A^1 = \tilde{A}$.

In fact we can exploit more the definition of a $p\underline{CCC}$-category. Notice that for every object $A$, $1 \times A$ is isomorphic to $A$. Therefore we have the following isomorphisms:

$$hom_{\underline{C_T}}(C \times B, \tilde{A}) \simeq hom_{\underline{C}}(1 \times (C \times B), A)$$

$$\simeq hom_{\underline{C}}(C \times B, A) \simeq hom_{\underline{C_T}}(C, A^B).$$

Complete objects in a $p\underline{CCC}$-category $\underline{C}$ satisfy the following:

2.6 Lemma. 1 is a complete object. Complete objects are closed under products and for every $\underline{C}$-objects $A, B$ $A^B$ is a complete object.

Proof. We will use the characterization of complete objects.

Indeed, 1 is a complete object because $!_B : B \to 1$ is maximum in $hom_{\underline{C}}(B, 1)$. Suppose now, $A, B$ are complete objects and take any $\underline{C}$-morphism $f : C \to A \times B$ then making $f_A = \pi_{A,B}f : C \to A, f_B = \pi'_{A,B}f : C \to B$; by the hypothesis, there are total morphisms $f_A^-, f_B^-$ such that $f_A \leq f_A^-$ and $f_B \leq f_B^-$. Notice that

$$f = \langle f_A, f_B\rangle$$

and $$f \leq \langle f_A^-, f_B^-\rangle.$$

Clearly, $\langle f_A^-, f_B^-\rangle$ is a total morphism.

Finally, if $A, B$ are arbitrary objects of $\underline{C}$ and $f : C \to A^B$ is any $\underline{C}$-morphism, then we want a total morphism $f^- : C \to A^B$; i.e., a $\underline{C}$-morphism $g : C \times B \to A$.

From $f : C \to A^B$, we have: $f \times 1_B : C \times B \to A^B \times B$.Call

$$ev_A = \lambda^{-1}(1_{A^B}) : A^B \times B \to A$$

then we get $g = ev_A(f \times 1_B) : C \times B \to A$. Define, $f^- = \lambda(g) : C \to A^B$; then, we have

$$f \leq \lambda(g)$$

simply because $\lambda(g) = \lambda(ev_A(f \times 1_B)) \leq \lambda(ev_A)f = f$.

In [Cu.Ob.], it was stated that the full subcetegory $\underline{C}_{TL}$ of $\underline{C}_T$ which has as objects 1 and $\tilde{A}$ for all $\underline{C}$-object $A$, is Cartesian Closed. The problem is that there is no good reason for $\tilde{A}^{\tilde{B}}$ be of the form $\tilde{C}$ for some $\underline{C}$-object $C$.

2.7 Proposition. If $\underline{C}$- is a partial cartesian closed category then the full subcategory $\underline{CC}_T$ of $\underline{C}_T$ whose objects are complete objects of $\underline{C}$, is a Weak cartesian closed category.

In [L.S.] ( see exercise 2, p.97) , the notion of a weak $\underline{C}$-monoid is introduced. In order to be more clear we will define a weak cartesian closed category.

2.8 Definition. A weak cartesian closed category $\underline{C}$, is a category with the following structure:

i. A terminal object 1

ii. For every objects $A, B$, an object $A \times B$ together with morphisms $\pi_{A,B} : A \times B \to B, \pi'_{A,B} : A \times B \to B$.

iii. For every objects $A, B$, an object $A^B$, and an arrow $ev_{A,B} : A^B \times B \to A$.

These data satisfy the following conditions:

a. For all $f : C \to A, g : C \to B$, there is an arrow

$$\langle f, g, \rangle : C \to A \times B.$$

b. For any $k : C \times B \to A$ there is an arrow $k^* : C \to A^B$.

satisfying the following equations.

1. $\pi_{A,B}\langle f, g\rangle = f$.
2. $\pi'_{A,B}\langle f, g\rangle = g$.
3. For all $h : D \to C, \langle f, g\rangle h = \langle fh, gh\rangle$.
4. $ev_{A,B}\langle k^* f, g\rangle = k\langle f, g\rangle$.
5. $k^* f = (k\langle f\pi_{A,B}, \pi'_{A,B}\rangle)^*$.

We will prove now the proposition 2.7.

Proof of 2.7 .

The only thing we need to show is the weak exponentiation. If $A, B$ are arbitrary $\underline{CC}_T$-objects, then by the lemma 2.6, $A^B$ is a complete object. We define first the evaluation morphism.

We already have a $\underline{C}$-arrow $ev_A : A^B \times B \to A$: since $A$ is a complete object, there is a total arrow

$$ev_{A,B} : A^B \times B \to A$$

such that $ev_A \leq ev_{A,B}$.

Given any $CC_T$-morphism $f : C \times B \to A$, we get a $CC_T$-morphism $f^* = \lambda(f) : C \to A^B$; we will show that

$$ev_{A,B}\langle f^* \pi_{C,B}, \pi'_{C,B}\rangle = f.$$

Since $\lambda(f), \pi_{C,B}, \pi'_{C,B}$ , are total morphisms we have: $\lambda(ev_A\langle f^*\pi_{C,B}, \pi'_{C,B}\rangle)f^* = \lambda(ev_A)f^* = \lambda(f)$ . Therefore $f = ev_A\langle f^*\pi_{C,B}, \pi'_{C,B}\rangle$. Using $ev_A \leq ev_{A,B}$ we get $f = ev_A\langle f^*\pi_{C,B}, \pi'_{C,B}\rangle \leq ev_{A,B}\langle f^*\pi_{C,B}, \pi'_{C,B}\rangle$ ; since $f$ is total we get:

$$ev_{A,B}\langle f^*\pi_{C,B}, \pi'_{C,B}\rangle = f.$$

From this equality, 2.8.4 and 2.8.5 follow easily, hence $\underline{CC}_T$ is a weak cartesian closed category.

How can we get a cartesian closed category from $\underline{CC}_T$ ? Using the Karoubi envelope of a category (see [L.S.], p.100), we can prove that the Karoubi envelope of $\underline{CC}_T$, denoted by $K(\underline{CC}_T)$, is a cartesian closed category. In fact we can state a more general result; if $\underline{C}$ is a weak cartesian closed category then $K(\underline{C})$ is cartesian closed. This result is stated for weak $\underline{C}$-monoids in [L.S.] ( see exercise 3 p.100).

**2.9 Proposition** The Karoubi envelope of a weak cartesian closed category $\underline{C}$ is cartesian closed.

Proof. Recall that the objects of $K(\underline{C})$ are idempotents $f_A : A \to A$ and the morphisms are $\underline{C}$-arrows $\phi : A \to B$ such that $f_B\phi f_A = \phi$. We will show first that $K(\underline{C})$ is cartesian.

Clearly, $K(\underline{C})$ has a terminal object. Now, if $f_A, f_B$ are to arbitrary objects of $K(\underline{C})$ the the product is:

$$f_A \times f_B = \langle f_A\pi_{A,B}, f_B\pi'_{A,B}\rangle.$$

Using 2.8 it is easy to see that $f_A \times f_B$ is an idempotent. The projections are $f_A\pi_{A,B} : f_A \times f_B \to f_A$ and $f_B\pi'_{A,B} : f_A \times f_B \to f_B$. We check, for instance that $f_A\pi_{A,B}$ is a morphism.

$$(f_A\pi_{A,B})(f_A \times f_B) = f_Af_A\pi_{A,B} = f_A\pi_{A,B}.$$

If $\psi : f_C \to f_A$ and $\phi : f_C \to f_B$ are two $K(\underline{C})$-morphisms then $\langle\psi, \pi\rangle : f_C \to f_A \times f_B$ is a $K(\underline{C})$-morphism because

$$\langle\psi, \phi\rangle f_C = \langle\psi f_C, \phi f_C\rangle$$
$$= \langle\psi, \phi\rangle.$$

Using 2.8, it is easy to see that $\langle\psi, \phi\rangle$ is unique.

Finally, $K(\underline{C})$ is cartesian closed. If $f_A, f_B$ are two arbitrary $K(\underline{C})$-objects $f_A^{f_B}$ is an idempotent and $f_Aev_{A,B}(1_{AB} \times f_B) : f_A^{f_B} \times f_B \to f_A$ is a morphism (using again 2.8). Given $h : f_C \times f_B \to f_A$, $h^* : f_C \to f_A^{f_B}$ satisfies:

i. $f_A^{f_B}h^*f_C$
$=(f_Aev_{C,B}\langle h^*\pi_{C,B}, f_B\pi'_{C,B}\rangle))^*f_C$
$=(f_Ah(1_C \times f_B))^*f_C.$
$(f_Ah(1_C \times f_B)(f_C \times 1_B))^*.$
$h^*.$

hence, $h^*$ is a $K(\underline{C})$-morphism.

ii. $f_Aev_{A,B}(1 \times f_B)(h^* \times 1_B)(f_C \times f_B).$
$=f_Aev_{A,B}(h^* \times 1_B)(f_C \times f_B).$
$f_Ah(f_C \times f_B) = h.$
iii. $h^*$ is unique.

If $k : f_C \to f_A^{f_B}$ satisfies also $f_A ev_{A,B}(1 \times f_B)(k \times 1_B)(f_C \times f_B) = h$. then since $k = f_A^{f_B} k f_C$ we have

$$k = (f_A ev_{A,B}(k f_A \times f_B))^* = h^*.$$

Therefore $K(\underline{C})$ is a cartesian closed category.

## 3. When $C_T$ is cartesian closed ?

If $\underline{C}$ is a $p\underline{CCC}$-category, when $\underline{C}_T$, the category of total maps is cartesian closed? We will give an answer to this but first we will make the following observation.

3.1 Lemma. If $\underline{C}$ is a partial category with liftings then the diagram

$$A \xrightarrow{\eta_A} \tilde{A} \underset{\tilde{\eta}_A}{\overset{\eta_{\tilde{A}}}{\rightrightarrows}} \tilde{\tilde{A}}$$

is a $\underline{C}_T$-equalizer.

Proof. The diagram commutes because

$$\eta_{\tilde{A}}\eta_A = \sigma_{\tilde{A},A}(1_{\tilde{A}})\eta_A$$

$$\tilde{\eta}_A\eta_A = \sigma_{\tilde{A},\tilde{A}}(\eta_A ex_A)\eta_A =$$

$$= \sigma_{A,\tilde{A}}(\eta_A).$$

hence $\eta_{\tilde{A}}\eta_A = \tilde{\eta}_A\eta_A$.

If $f : C \to \tilde{A}$ is a $\underline{C}_T$-morphism such that $\eta_{\tilde{A}} f = \tilde{\eta}_A f$ then

$$\eta_{\tilde{A}} f = \sigma_{\tilde{A},\tilde{A}}(1_{\tilde{A}}) f = \sigma_{C,\tilde{A}}(f)$$

and $\tilde{\eta}_A f = \sigma_{\tilde{A},\tilde{A}}(\eta_A ex_A) f = \sigma_{C,\tilde{A}}(\eta_A ex_A f)$; hence $f = \eta_A(ex_A f)$. Since $f$ is a total morphism then $ex_A f$ is also a total morphism.

By the remark after 2.5, we already know that for any objects $\tilde{A}, B, (\tilde{A})^B$ exists in $\underline{C}_T$. Using the lemma 3.1 we can prove the following:

3.2 Proposition If $\underline{C}_T$ has equalizers then $\underline{C}_T$ is a cartesian category.

Proof. Let $A, B$ any objects of $\underline{C}_T$ considerthe following two $\underline{C}$-arrows:

$$\{\eta_{\tilde{A}}\}^B : (\tilde{A})^B \to (\tilde{\tilde{A}})^B.$$

$$\{\tilde{\eta}_A\}^B : (\tilde{A})^B \to (\tilde{\tilde{A}})^B.$$

Define $A^B$ as the equalizer of these two arrows:

$$A^B \xrightarrow{\epsilon} (\tilde{A})^B \underset{(\tilde{\eta}_A)^B}{\overset{(\eta_{\tilde{A}})^B}{\rightrightarrows}} (\tilde{\tilde{A}})^B$$

We can form the following commutative diagram:

Then we have an arrow $ev_{A,B} : A^B \times B \to A$. Now, given any morphism $f : C \times B \to A$ in $\underline{C}_T$ , we get a $\underline{C}_T$-morphism $f^\dagger : C \times B \to \tilde{A}$ and therefore $(f^\dagger)^\star : C \to (\tilde{A})^B$ equalizes $(\eta_{\tilde{A}})^B$ and $(\tilde{\eta}_A)^B$ because:

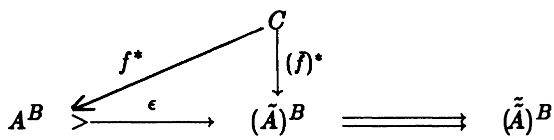
$$ev_{\tilde{\tilde{A}},B}((\eta_{\tilde{A}})^B)((f^\dagger)^\star \times 1_B)$$

$$= \eta_{\tilde{A}} ev_{\tilde{A},B}((f^\dagger)^\star \times 1_B) = \eta_{\tilde{A}} f^\dagger.$$

and

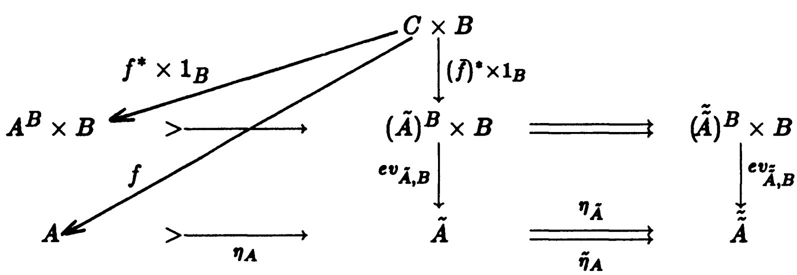
$$ev_{\tilde{\tilde{A}},B}((\tilde{\eta}_A)^B (f^\dagger)^\star \times 1_B)) == \tilde{\eta}_A ev_{\tilde{A},B}((f^\dagger)^\star \times 1_B) = \eta_{\tilde{A}} f^\dagger.$$

it is easy to see that $\eta_{\tilde{A}} f^\dagger = \tilde{\eta}_A f^\dagger$ using $ex_A f^\dagger = f$. Therefore there is an arrow $f^\star : C \to A^B$ making the diagram



commute.

Clearly, $ev_{A,B}(f^\star \times B) = f$, because we can form the following commutative diagram



Now,given any $h : C \to A^B$ in $\underline{C}_T, (ev_{A,B}(h \times 1_B))^\star = h$ if and only if $\eta_{A^B}(ev_{A,B}(h \times 1_B))^\star = \eta_{A^B} h$; Where the arrow $\eta_{A^B}(ev_{A,B}(h \times 1_B))^\star$ is given by:

$(\overline{ev_{A,B}(h \times 1_B)})^\star$ and $\overline{ev_{A,B}(h \times 1_B)} = \overline{ev_{A,B}}(h \times 1_B)$; Moreover, $\overline{ev_{A,B}} = \eta_A ev_{A,B}$ as is easily seen; hence

$$\eta_{A^B}(ev_{A,B}(h \times 1_B))^\star = \eta_A ev_{A,B}(h \times 1_B)$$

$$= ev_{\tilde{A},B}(\eta_{A^B} \times 1_B)(h \times 1_B) = ev_{\tilde{A},B}(\eta_{A^B} h \times 1_B)$$

Therefore, $(ev_{A,B}(h \times 1_B))^\star = (ev_{\tilde{A},B}(\eta_{A^B} h \times 1_B))^\star = \eta_{A^B} h$ and we prove the assertion.

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