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Deformations and Quantum Statistical Mechanics

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Abstract

From 1978, Flato, Fronsdal and myself have studied deformations of the Poisson bracket structure of Classical Mechanics and related associative algebra structures. With the corresponding notion of star-product, we have thus obtained a geometrical approach to quantization (deformation parameter $\nu = \hbar/2i$). In this talk, I will sketch how a further deformation–based on conformal symplectic structure–allows one to include another deformation parameter $\beta = (1/kT)$ adapted to Statistical Mechanics. Topics of the talk are : star-products and quantization, deformations of local associative algebras, Statistical Mechanics in terms of star-products, classical and quantum (KMS)-conditions.

1 Notion of star–product.

Let W be a Banach differentiable manifold (of finite or infinite dimension) on which there exist partitions of the unity. Denote by N the space of the real–valued infinitely differentiable functions on W, a space that can be eventually restricted. In the following, we use as technical instrument *the Schouten bracket* on the skew-symmetrical contravariant tensors or *p-tensors*.

(a) *A Poisson manifold* $(W, \wedge)$ is a manifold endowed with a skew symmetrical contravariant 2-tensor (or 2-tensor) satisfying, in the sense of the Schouten bracket,

$$[\wedge, \wedge] = 0 \tag{1.1}$$

$\wedge$ defines on W *a Poisson bracket* satisfying the Jacobi identity

$$\{u, v\} = i(\wedge)(du \wedge dv) = P(u, v) \quad (u, v \in N) \tag{1.2}$$

where P is the Poisson bidifferential operator. If $u \in N$, the corresponding Hamiltonian vector field is $[\wedge, u]; (W, \wedge)$ is *symplectic* if $\wedge$ is *non-degenerate* that is if $[\wedge, u] = 0$ implies $du = 0$ for any $u \in N$.

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Let $E(N;\nu)$ be the space of the formal functions in $\nu \in \mathbb{C}$, with coefficients in N; ν is the deformation parameter (for quantum Mechanics $\nu = \hbar/2i$). Consider a formal associative deformation of the associative (and commutative) composition law on N given by the usual product of functions. Such a deformation is strictly connected with the Hochschild cohomology of this composition law. More precisely, we say that we have a star-product $*_\nu$ on the Poisson manifold (W, Λ) if we have

$$u *_\nu v = uv + \nu P(u,v) + \sum_{r=2}^{\infty} \nu^r C_r(u,v) \quad (u, v \in N) \tag{1.3}$$

where the C_r are bidifferential operators–or 2-cochains–satisfying the following requirements [3]

(i) *The canonical extension of* (1.3) *to* $E(N;\nu)$ *is associative; we have*

$$(u *_\nu v) *_\nu w = u *_\nu (v *_\nu w) \qquad (\text{for } u, v, w, \in N \textit{ and so } \in E(N;\nu))$$

(ii) *The* C_r *are even in* u, v *if* r *is even, odd if* r *is odd (parity condition), that is*

$$u *_{-\nu} v = v *_\nu u \qquad (u, v \in N) \tag{1.4}$$

(iii) *The* C_r *are null on the constants, so that*

$$1 *_\nu u = u *_\nu 1 = u \tag{1.5}$$

If we skew symmetrize $u *_\nu v$ in u, v we obtain a formal deformation of the Poisson Lie algebra : for $[u,v]_{*_\nu} = [u,v]_\nu = (2\nu)^{-1}(u *_\nu v - v *_\nu u)$ we have

$$[u,v]_\nu = P(u,v) + \sum_{r=1}^{\infty} \nu^{2r} C_{2r+1}(u,v) \tag{1.6}$$

which defines on $E(N;\nu)$ a structure of Lie algebra.

It is known (M. de Wilde and P. Lecomte [4]) that such a star-product exists always on a finite dimensional symplectic manifold. If (W, Λ) is a finite-dimensional or infinite dimensional symplectic manifold, we suppose that there exists on (W, Λ) a $*_\nu$-product defined on a suitable algebra of functions. We have then *a starred Poisson manifold* $(W, \Lambda, *_\nu)$.

2 Star–products and quantization.

(a) Consider $W = \mathbb{R}^{2n}$ endowed with its usual symplectic structure Λ; $(\mathbb{R}^{2n}, \Lambda)$ admits a natural star–product, the so–called *Moyal product* given by exp $((\hbar/2i)P)(u,v)$. The Weyl–Wigner quantization rule maps functions to operators on $L^2(R^n)$ and Moyal products to operator products, the mapping be in one–to–one for a suitable N. The Weyl quantization rule is the historical origin of the Moyal product [5].

Looking at quantization as a deformation of the composition laws of functions, that is as the choice of a star–product, it is then possible to develop quantum theories in terms of the star–product only, in a completely autonomous manner, even if a Weyl–type of quantization rule does not appear. In particular a spectral theory can be developed using only the deformed products [3].

We can define the p^{th} star–power of a function, for example of the classical Hamiltonian H; we denote this power by $H^{(\ast)p}$. It is easy to prove by induction on p that $H^{(\ast)p}$ is always an even function of ν. We can set formally :

$$\mathrm{Exp}_{*}\left(\frac{i}{\hbar}\, tH\right) = \sum_{p=0}^{\infty}\left(\frac{i}{\hbar}\, t\right)^{p} \frac{H^{(\ast)p}}{p!} \tag{2.1}$$

For usual star–products and Hamiltonians, the right member of (2.1) defines a distribution on W, which plays here the role of the operator $e^{|i|\hbar t H}$ and governs the evolution of the system. The spectrum of H is the spectrum of $\mathrm{Exp}_{*}\left(\frac{i}{\hbar} tH\right)$ in the sense of Schwartz. In a similar way, we can define $\mathrm{Exp}_{*}(\beta H)$, where β is a suitable real parameter.

Generally speaking, when we adopt the point of view of the mathematical analysis, we take $\nu = \hbar/2i$ and denote the star–product by $*$ for suitable functional spaces.

(b) Suppose that $\dim W = 2n, (W, \Lambda)$ being symplectic. The manifold admits a measure $\tilde{\eta} = (2\pi\hbar)^{-n}\eta$, where η is the symplectic volume element of (W, Λ) (normalized Liouville measure). Let N^c be the complexified space of N; the star–product can be extended to N^c in a natural way. Suppose that our star–product satisfies the following requirements.

(i) The star–product is *non degenerate* : if $u \in N^c$, $\bar{u} * u = 0$ implies $u = 0$.

(ii) If u, v are two functions with compact supports or suitable asymptotic conditions

$$\int_W [u,v]_* \tilde{\eta} = 0 \qquad (2.2)$$

It is easy to construct on $(W, \wedge)$ star-products satisfying these requirements. In particular the Moyal product on $\mathbb{R}^{2n}$ admits these properties.

The expectation value associated with an observable a and a state ρ given by real-valued functions is defined by

$$\int (a * \rho)\tilde{\eta} = \int (\rho * a)\tilde{\eta}$$

This formula is an extension of a classical formula of Wigner.

3 Deformations of local associative algebras–Extensible infinitesimal deformations.

Let W be a smooth finite-dimensional manifold and $N = N(W)$ the space of real-valued infinitely differentiable functions on W.

(a) *A local associative algebra* $(N, \Box)$ is an associative algebra on N such that

$$\operatorname{Supp}(u \Box v) \subset \operatorname{Supp} u \cap \operatorname{Supp} v \qquad (u, v \in N)$$

It is known (Rubio [6]) that such an algebra is necessarily given by

$$u \Box v = f u v \qquad (3.1)$$

where f is a fixed element of N; the algebra is necessarily commutative. If $(N, \Box)$ admits a unit element e, the function f is $\neq 0$ everywhere and $e = f^{-1}$. In the following $(N, \Box)$ *is the local associative algebra given by* (3.1) *where* f *is* $\neq 0$ *everywhere.*

Introduce the Hochschild cohomology of $(N, \Box)$ with values in N. A p-cochain C is a mapping of $N \times N \times \ldots \times N$ (p factors) into N and the corresponding standard coboundary operator is denoted by $\tilde{\partial}$ [1]. In particular if C is a two-cochain, we have

$$\begin{aligned} \tilde{\partial} C(u,v,w) &= u \Box C(v,w) - C(u \Box v, w) + C(u, v \Box w) \\ &- C(u,v) \Box w \quad (u,v,w \in N) \end{aligned} \qquad (3.2)$$

Consider a formal associative deformation of $(N, \Box)$ defined by the map $N \times N \to E(N, \nu)$ given by

$$u *_\nu^f v = \sum_{r=0}^{\infty} \nu^r C_r^f(u,v) = fuv + \sum_{r=1}^{\infty} \nu^r C_r^f(u,v) \qquad (3.3)$$

where the $C_r^f (r \geq 1)$ are local 2–cochains such that

Parity condition : C_r^f *is even if* r *is even, odd if* r *is odd:* $u *_{-\nu}^f v = v *_\nu^f u$.

If (3.3) is limited to the order q in ν, we say that we have a deformation of order q, if the associativity identity is satisfied up to the order $(q+1)$. In particular a deformation of order 1 is called *an infinitesimal deformation.*

(b) Consider the deformation of order 2 of $(N, \square)$

$$u *_\nu^f v = fuv + \nu C_1^f(u,v) + \nu^2 C_2^f(u,v) \tag{3.4}$$

where the two–cochains C_1^f, C_2^f are local, C_1^f being odd and C_2^f even. It is easy to verify that we have [2]

$$\tilde{\partial} C_1^f = 0 \tag{3.5}$$

and

$$\tilde{\partial} C_2^f = \tilde{E}_2 \tag{3.6}$$

where $\tilde{E}_2$ is the three–cochain given by

$$\tilde{E}_2(u,v,w) = C_1^f(C_1^f(u,v),w) - C_1^f(u, C_1^f(v,w))$$

For an arbitrary three–cochain B, we set

$$(\hat{\sum} B)(u,v,w) = B(u,v,w) - B(v,u,w) - B(u,w,v)$$

It is easy to see that if C is an even two–cochain, we have $\hat{\sum}\tilde{\partial} C = 0$. It follows from (3.6) that $\hat{\sum}\hat{E}_2 = 0$, that is

$$S\, C_1^f(C_1^f(u,v),w) = 0 \tag{3.7}$$

where S is the summation over cyclic permutations. Equation (3.7) expresses that C_1^f defines on N the bracket of *a local Lie algebra.* It is known then (Kirillov [10]) that there exist on W a 2–tensor Λ^f and a vector E^f such that $C_1^f = P^f$, where

$$P^f(u,v) = i(\Lambda^f)(du \wedge dv) + i(E^f)(udv - vdu) \tag{3.8}$$

P^f satisfying the Jacobi identity, we have in terms of Schouten brackets

$$[\Lambda^f, \Lambda^f] = 2E^f \wedge \Lambda^f \qquad [E^f, \Lambda^f] = \mathcal{L}(E^f)\Lambda^f = 0 \tag{3.9}$$

where $\mathcal{L}(\ )$ is the operator of Lie derivative. We see that if $fuv + \nu C_1^f$ is an infinitesimal of formation of $(N, \Box)$ extensible to the order 2, $C_1^f = P^f$, where P^f is the above bracket of a Lie algebra. It is easy to see that if $\tilde{\partial} P^f = 0$, we have $i(E^f)df = 0$ and so $E^f = f^{-1}[\Lambda^f, f]$ [2].

If we set $\Lambda^f = f\Lambda$. (3.9) is translated by $[\Lambda, \Lambda] = 0, E^f = [\Lambda, f]; \Lambda$ defines on W a Poisson structure and E^f is the Hamiltonian vector field associated with f. We obtain [2].

Proposition 1 *Each extensible infinitesimal deformation of* $(N, \Box)$ *is given by*

$$u *_\nu^f v = fuv + \nu P^f(u, v)$$

where P^f *is the bracket of the conformal Poisson structure deduced from a Poisson structure* Λ *by the conformal factor* f.

(c) Let W be a finite or infinite-dimensional Banach manifold. We consider on $N = N(W)$ an associative algebra $(N, \Box)$ given by $u, v \in N \to fuv \in N$ where $f \in N$ is $\neq 0$ everywhere. Consider an associative deformation of $(N, \Box)$ given by

$$u *_\nu^f \nu = fuv + \nu P^f(u, v) + \sum_{r=2}^{\infty} \nu^r C_r^f(u, v) \tag{3.10}$$

where the 2-cochains C_r^f are differentiable, satisfy the parity condition and where P^f is the bracket of the conformal Poisson structure (Λ^f, E^f) deduced from a Poisson structure by the conformal factor f. *If* (3.10) *admits a unit element* e_ν *(necessarily even in* ν*) we say that* (3.10) *is a* f*-star product on* W. We have proved [2].

Theorem 1 *If (3.10) is a* f*-star product, there is a star-product* $*_\nu$ *of* (W, Λ) *and an even element* f_ν *of* $E(N; \nu)$ *satisfying* $f_0 = f$ *such that (3.10) is given by*

$$u *_\nu^f \nu = u *_\nu f_v *_\nu v \tag{3.11}$$

Conversely, for an arbitrary choice of f_v *and* $*_\nu$*, (3.11) defines a* f*-star-product admitting the unit element* $f_\nu^{(\bullet)-1}$.

The existence or equivalence problems concerning the f-star-products can be thus reduced to the same problems concerning the star-products.

4 Statistical mechanics.

Let $(W, \Lambda *)$ be a starred symplectic Banach manifold which is the phase space of the considered physical system. On this manifold, let $(A, *)$

be an algebra of observables contained in N (the algebra of the so-called quasi local observables) and containing the function 1 (which is the unit element of the algebra). A state ω is described by a continuous linear functional on A and for $a \in A$, $\omega(a)$ is the estimation value corresponding to the observable a and to the state ω.

Let H be the Hamiltonian of our physical system; H does not belong necessarily to A. The quantum dynamics of the system is given by the one-parameter group of automorphisms of A

$$\alpha_t(u) = \operatorname{Exp}_*((i/\hbar)\, tH) * u * \operatorname{Exp}_*(-(i/\hbar)\, tH) \tag{4.1}$$

If $\beta = 1/kT$ we suppose that for every real $\tau (0 \leq \tau \leq 1)$

$$\alpha_{t+i\tau\beta}(u) = \operatorname{Exp}_*((i/\hbar)(t + i\hbar\tau\beta)H) * u * \operatorname{Exp}_*(-(i/\hbar)(t + i\hbar\tau\beta)H) \tag{4.2}$$

is an automorphism of A

We note that

$$\alpha_{i\hbar\tau\beta}(u) = \operatorname{Exp}_*(-\tau\beta H) * u * \operatorname{Exp}_*(\tau\beta H)$$

and that $\operatorname{Exp}_*(\tau\beta H)$ or $\operatorname{Exp}_*(-\tau\beta H)$ satisfy the parity assumption in $\nu = \hbar/2i$; $\operatorname{Exp}_*[-\beta H]$ for example describes the so called-*canonical ensemble.* We set

$$f_\beta = \operatorname{Exp}_*(\tau\beta H) \tag{4.3}$$

and introduce eventually, in the following part, the associative product

$$u \tilde{*}_\beta v = u * f_\beta * v \tag{4.4}$$

5 Classical (KMS)– condition.

(a) The (KMS)-condition is an important and very nice condition characterising in Statistical Mechanics the Gibbs states. Haag-Kastler and Araki have studied, under strong analytical assumptions, the physical origin of this condition ([7] and [8]). Haag and Kastler have proposed to consider the following as the three defining properties of an equilibrium state ω.

(i) *Stationarity of the state* : $\omega(\alpha_t(a)) = \omega(a)$ for $a \in A$

(ii) *Relative purity of the stationary state* ω : ω cannot be split into a convex combination of different stationary states

(iii) *Stability under local perturbations of the Dynamics* (that is of H). Under suitable analytic assumptions, Haag–Kastler et al have proved that, in the quantum case, the stability condition for the stationary states is translated by the (KMS)–condition. This in turn implies that it is a Gibbs state. In 1976 Aizenmann et al [9] have proved the same for the classical Statistical Mechanics under other analytic assumptions.

We see that the stability implies very strong constraints on ω. The quantum (KMS)–condition can be written in terms of star–products by translation of the usual operator condition : ω satisfies the (KMS)–condition if for any observables $a, b \in A$

$$\omega(a * b) = \omega(b * \alpha_{i\hbar\beta}(a)) \tag{5.1}$$

(b) From the classical limit of Equation (5.1). We note that for $\nu = \hbar/2i$, we have up to terms of order ≥ 2 in ν

$$\alpha_{i\hbar\beta}(a) \simeq a - (\hbar/i)\beta P(H, a)$$

It follows from (5.1) that

$$\omega((i/\hbar)(a * b - b * a)) \simeq -\beta\omega(bP(H, a))$$

We obtain thus for the classical (KMS)–condition

$$\omega(P(a, b)) = -\beta\omega(bP(H, a)) \qquad \beta > 0 \tag{5.2}$$

that is the form obtained by Aizenmann et al up to the notations [9].

(c) It is easy to obtain an interesting and geometric equivalent form. By skew symmetrization of (5.2) in a, b, (5.2) implies

$$\omega(P(a, b)) = (\beta/2)\omega(aP(H, b) - bP(H, a)) \tag{5.3}$$

Conversely assume (5.3). If we take $b = 1$, we have $\omega(P(H, a)) = 0$ for any a, relation which expresses that ω is stationary. For arbitrary observables a, b, we have then $\omega(P(H, ab)) = 0$ and (5.3) can be written

$$\omega(P(a, b)) = (\beta/2)\,\omega\,(aP(H, b) - bP(H, a) - P(H, ab))$$

which, after simplifications is nothing other but (5.2). Therefore (5.3) is equivalent with (5.2).

Take $f = f_\beta = e^{-(\beta/2)H}$ in (3.8). We have $\wedge_\beta = e^{-(\beta/2)H}\wedge$ and $E_\beta = -\frac{\beta}{2}e^{-(\beta/2)H}[\wedge, H]$.

The bracket (3.8) takes the form

$$P^{f\beta}(u,v) = [u,v]_\beta = e^{-(\beta/2)H}(P(u,v) - (\beta/2)uP(H,v) + (\beta/2)vP/H,u))$$

It follows that (5.3) can be written.

$$\omega[e^{(\beta/2)H}[a,b]_\beta] = 0 \quad \text{for} \quad f_\beta = e^{-(\beta/2)H} \tag{5.4}$$

We have [1]

Theorem 2 *The classical (KMS)-condition is equivalent to (5.4) where $[a,b]_\beta$ is the bracket associated with the conformal symplectic structure corresponding to the conformal factor $f_\beta = e^{-(\beta/2)H}$.*

6 Quantum (KMS)–condition.

(a) Come back to the quantum (KMS)-condition (5.1). We prove first the following lemma.

Lemma 1 *If ω is a stationary state, we have for $0 \leq \tau \leq 1$*

$$\omega(\alpha_{i\hbar\tau\beta}(a) - a) = 0 \tag{6.1}$$

In fact, if ω is stationary $\omega(\alpha_t(a)) = \omega(a)$ for all $a \in A$ and it follows by differentiation in t for $t = 0$ that $\omega([H,a]_*) = 0$. If we substitute to a the observable $\alpha_{i\hbar\tau\beta}(a)$ we obtain

$$\omega([H, \alpha_{i\hbar\tau\beta}(a)]_*) = 0 \tag{6.2}$$

Consider for $0 \leq t \leq 1$ the function

$$g(\tau) = \omega(\alpha_{i\hbar\tau\beta}(a) - a) \qquad 0 \leq \tau \leq 1$$

Differentiating in τ we obtain

$$dg(\tau)/dt = i\hbar\beta\,\omega([H, \alpha_{i\hbar\tau\beta}(a)]_*) = 0$$

We have $g(\tau) = \text{const.}$; take $\tau = 0$, we obtain $g(0) = 0$. We have $g(\tau) = 0$ for $0 \leq \tau \leq 1$ and (6.1) is proved.

(b) Take $\tau = 1/2$: we set $f_\beta = \text{Exp}_*(-(\beta/2)H)$ and introduce the automorphism σ of A given by

$$\sigma a = \alpha_{i(\hbar/2)\beta}(a) = f_\beta * a * f_\beta^{(\bullet)-1}$$

(6.1) implies $\omega(\sigma a) = \omega(a)$ for all $a \in A$; (5.1) can be written

$$\omega(a * b) = \omega(b * \sigma^2 a) \qquad \text{for all} \quad a, b \in A \tag{6.3}$$

In (6.3) substitute to b the observable σb. We have, according to (6.3) :

$$\omega(a * \sigma b) = \omega(\sigma(b * \sigma a)) = \omega(b * \sigma a)$$

that is

$$\omega(a * \sigma b - b * \sigma a) = 0 \tag{6.4}$$

Introduce

$$\begin{aligned} [a,b]_{\tilde{*}\beta} * f_\beta^{(*)-1} &= (i/\hbar)(a * f_\beta * b * f_\beta^{(*)-1} - b * f_\beta * a * f_\beta^{(*)-1}) \\ &= (i/\hbar)(a * \sigma b - b * \sigma a) \end{aligned} \tag{6.5}$$

Note that, since σ is an automorphism of A, $[a,b]_{\tilde{*}\beta} * f_\beta^{(*)-1}$ and $f_\beta^{(*)-1} * [a,b]_{\tilde{*}\beta} = \sigma^{-1}([a,b]_{\tilde{*}\beta} * f_\beta^{(*)-1})$ belong to A even if f_β cannot be defined in A. If follows from (6.4), (6.5) that

$$\omega(\mathrm{Exp}_*((\beta/2)H) * [a,b]_{\tilde{*}\beta}) = 0$$

The converse is evident, we have

Theorem 3 *On a starred manifold $(w, \wedge, *)$, the quantum (KMS)-condition is equivlent to*

$$\omega(\mathrm{Exp}_*((\beta/2)H) * [a,b]_{\tilde{*}\beta}) = 0 \tag{6.6}$$

where $[a,b]_{\tilde{*}\beta}$ is the bracket deduced from the associative product

$$\tilde{*}_\beta = *\mathrm{Exp}_*(-(\beta/2)H) * .$$

In terms of operators (6.6) takes the form

$$\omega(e^{(\beta/2)H} A e^{-(\beta/2)H} B - e^{(\beta/2)H} B\, e^{-(\beta/2)H} A) = 0 \tag{6.7}$$

and it is possible to prove directly this equivalence.

What we find interesting from the deformation point of view is how we are led in a very natural way to introduce conformal symplectic geometry. We obtain thus simple and geometric expressions for the (KMS)–conditions (both classical and quantum); it just tells us that, up to a temperature–dependant conformal factor, the (KMS)–states should be those which see a Lie algebra of observables as *abelian*.

Note that the point of view of this talk is geometrical and physical. We do not consider the corresponding problems of Mathematical Analysis.

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