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Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications

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MATHEMATICS

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Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications

Proceedings of an AMS Special Session
held March 22–23, 1991
in Tampa, Florida

Donald St. P. Richards
Editor

American Mathematical Society
Providence, Rhode Island

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Preface

This volume is largely based on the lectures presented during a special session of the 865th meeting of the American Mathematical Society, convened in Tampa, Florida, during the period March 22 - 23, 1991. This special session, entitled "Hypergeometric functions on domains of positivity, Jack polynomials and applications," was centered on a branch of research initiated some forty years ago by Bochner. The initial impetus for Bochner's work came from questions in analytic number theory. It is remarkable that, since then, these hypergeometric functions have been found to be important in areas as diverse as combinatorics, harmonic analysis, molecular chemistry, multivariate statistics, partial differential equations, probability theory, representation theory and mathematical physics. In addition, the scope of these functions has been broadened considerably. While the initial investigations of these hypergeometric functions were carried out within the context of matrix spaces, the articles within this volume relate these functions to the study of domains of positivity and root systems.

At this stage, some brief, historical remarks are in order. Given a real, positive definite (symmetric) $n \times n$ matrix, Λ, with *integer* entries, let $r(\Lambda)$ denote the number of $k \times n$ matrices T, with integer matrices, and where $k \geq n$, such that $T'T = \Lambda$. A problem considered by Bochner (and others) is to investigate the asymptotic behavior of $r(\Lambda)$ as "$\Lambda \to \infty$." A natural approach to studying the asymptotic behavior of $r(\Lambda)$ is to study a generating function for $r(\Lambda)$. Thus if Z is also positive definite and $n \times n$, and $\operatorname{etr}(Z) \equiv \exp(\operatorname{tr} Z)$, define the theta function

$$\Theta(Z) = \sum_T \operatorname{etr}(-\pi T Z T').$$

It is not difficult to derive Jacobi's formula

$$\Theta(Z) = (\det Z)^{k/2}\, \Theta(Z^{-1}),$$

hence

$$\sum_\Lambda r(\Lambda)\ \operatorname{etr}(-\pi \Lambda Z) = (\det Z)^{k/2} \sum_\Lambda r(\Lambda)\ \operatorname{etr}(-\pi \Lambda Z^{-1}). \tag{1}$$

The formula (1) is, of course, a *modular relation*:

$$\sum_M a_M \operatorname{etr}(-MZ) = (\det Z)^{-\delta} \sum_\Lambda b_\Lambda \operatorname{etr}(-\Lambda Z^{-1}), \tag{2}$$

where $\delta \in \mathbb{C}$ with $\mathrm{Re}(\delta)$ sufficiently large. To extend (2) to more general modular relations of the form

$$\sum_M a_M f(M) = (\det Z)^{-\delta} \sum_\Lambda b_\Lambda g(\Lambda), \tag{3}$$

for functions f and g, Bochner defined the Bessel (or ${}_0F_1$ hypergeometric) function of matrix argument, by the integral equation

$$(\det Z)^{-\delta}\operatorname{etr}(-\Lambda Z^{-1}) = \int_{M>0} (\det M)^{\delta-(n+1)/2}\operatorname{etr}(-MZ)\,{}_0F_1(\delta; \Lambda M)\, dM. \tag{4}$$

Here $\{M > 0\}$ denotes the space of positive definite matrices, and dM is the corresponding Lebesgue measure. Of course, it must be verified that the function ${}_0F_1$ is well-defined by (4). On doing so, Bochner then proved that the modular relation (2) extends to (3), for certain classes of functions f, whenever g is the *Hankel transform* of f:

$$g(\Lambda) = \int_{M>0} f(M)\ {}_0F_1(\delta; \Lambda M)\ (\det M)^{\delta-(n+1)/2}\ dM. \tag{5}$$

If we write $g = \mathcal{H}_\delta f$ whenever (5) holds, then a more general theorem of Bochner (*Ann. Math.*, 1951) is the following.

THEOREM. (Bochner, 1951) *Suppose that R and S are positive Borel measures on $\{\Lambda > 0\}$ such that*

$$\int_{\Lambda>0} \operatorname{etr}(-\Lambda Z)\ R(d\lambda) = (\det Z)^{-\delta} \int_{M>0} \operatorname{etr}(-MZ^{-1})\ S(dM),$$

where $Z > 0$. If $g = \mathcal{H}_\delta f$ and f is completely monotone (the Laplace of a positive Borel measure on the cone of positive definite matrices) then

$$\int_{\Lambda>0} f(\Lambda) R(d\lambda) = (\det Z)^{-\delta} \int_{M>0} g(M) S(dM).$$

The proof of this result consists of writing the function f in the form

$$f(\Lambda) = \int_{\xi>0} \operatorname{etr}(-\Lambda\xi)\ \nu(d\xi),$$

where ν is a positive measure, and computing the Hankel transform of f by repeated interchanges of integration.

After Bochner's initial investigations, there was rapid development of the theory of hypergeometric functions of matrix argument and their more general relatives. In particular, Herz used the Laplace transform to define the general family of matrix argument functions, ${}_pF_q$; James and Constantine developed the zonal (spherical) polynomial series expansions for the class of ${}_pF_q$ hypergeometric functions, and defined the hypergeometric functions of two matrix arguments. The names and results of Gindikin, Jack, Macdonald, Muirhead, and others are now well-known in this area. The theory has found its most extensive applications in multivariate statistical analysis, notably in the area of "noncentral distribution theory."

As for the papers appearing here, I am pleased to note that they cover a broad range of applications of these hypergeometric functions. A common thread running through all the articles is the use of spherical functions, in the form of zonal or Jack polynomials or even as matrix entries for irreducible representations. The reader will find connections with mathematical physics, p-adic analysis, harmonic analysis, random walks, combinatorics, root systems, q-hypergeometric series, representation theory, random matrices, and operator theory. As important is the fact that they provide many references to the literature, and point towards open problems. Additional directions for future work can be found in the abstracts of talks which were presented at the special session but not included here because of the exigencies of time. In particular, I take all blame for the fact that applications to multivariate statistics, total positivity, and related topics will have to appear elsewhere.

Before closing, I do wish to draw the reader's attention to one type of open problem, with important implications for multivariate statistics and probability theory. Suppose that ${}_pF_q(X,Y)$ is a generalized hypergeometric function of two "matrix" arguments. In the classical matrix argument context, X and Y are real symmetric matrices; in the general Jack-polynomial context, $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ are n-dimensional vectors. It is of interest to statisticians to determine the nonnegativity of the functions

$$\frac{\partial^2}{\partial x_1 \partial x_2} \log[V(X)_p F_q(X,Y)]$$

and

$$\frac{\partial^2}{\partial x_1 \partial y_1} \log[V(X)V(Y)\ {}_pF_q(X,Y)]$$

where $V(X)$ is an anti-symmetric function of $x_1, \ldots, x_n$. In the case where we begin with hypergeometric functions defined on the cone of *Hermitian* positive definite matrices, in which case the Jack polynomials are the familiar Schur functions, much is known; in this case, it is no surprise that the appropriate choice for V is $V(X) = \prod_{1 \leq i < j \leq n}(x_i - x_j)$. An extensive study in the Hermitian case is currently in preparation (Chang, Peddada and Richards, 1992). But for no other situation do any results appear to be available. The reason for this paucity of results in the non-Hermitian cases is that only in the Hermitian case are explicit formulas for the general matrix argument ${}_pF_q$ functions available in terms of their classical counterparts (Gross and Richards, *Bull (N.S.) Amer. Math. Soc.*, 1991).

It is a great pleasure to acknowledge the support I received from the speakers at the special session. Bearing in mind that all speakers traveled at their own expense, the international flavor of the special session is more greatly appreciated. In particular, I note that the American, Canadian, Dutch, French, Jamaican and Russian schools were represented at the meeting. I thank the contributors for their patience during the processing of the volume. I also thank the referees for their prompt and accurate replies.

Finally, I thank Barbara Palmore who persevered, during difficult times, to convert all the contributions into a uniform format; John Stembridge who provided expert advice on $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX and a template for other articles; Donna Harmon and her production staff for their extreme patience; the American Mathematical Society for providing me with the opportunity to convene the special session and produce this volume; Ken Gross and Ray Kunze who encouraged me to organize the session; and last, but by no means, least, Mercedes, Chandra and Suzanne Richards who allowed me to neglect them during the final stages of the production process.

Donald St. P. Richards

Charlottesville

May, 1992

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Hypergeometric functions on domains of positivity, Jack polynomials, and applications
Donald St. P. Richards, Editor

This book is the first set of proceedings to be devoted entirely to the theory of hypergeometric functions defined on domains of positivity. Most of the scientific areas in which these functions now find applications—including analytic number theory, combinatorics, harmonic analysis, random walks, representation theory, and mathematical physics—are represented here. The volume is based largely on lectures presented at a Special Session at the AMS meeting in Tampa, Florida in March 1991, which was devoted to hypergeometric functions of matrix argument and to fostering communication among representatives of the diverse scientific areas in which these functions are utilized. Accessible to graduate students and others seeking an introduction to the state-of-the-art in this area, this book would be suitable as a text for advanced graduate seminar courses and contains many open problems.

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