The Reconstruction of Trees from Their Automorphism Groups

Matatyahu Rubin
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Summary

This work addresses the following question: "Find meaningful necessary and sufficient conditions for two trees to have isomorphic automorphism groups." To answer this question, we consider a more general type of structures, called unary trees.

**Definition:** (a) A poset \((T, <)\) is a tree, if for every \(t \in T, \{s \in T | s < t\}\) is linearly ordered by \(<\). \(M = (T, <, \{P_i\}_{i \in I})\) is a unary tree (U-tree), if \((T, <)\) is a tree, and for every \(i \in I, P_i \subseteq T\). Informally, A U-tree is a structure which is a tree together with a list of named subsets, each of which is required to be invariant under all automorphisms of the structure. So, for the above \(M, \text{Aut}(M) = \{f \in \text{Aut}(T, <) | (\forall i \in I)(f(P_i) = P_i)\}\).

(b) Let \(M\) be as in (a), then \(T(M)\) denotes \(\langle T, <, \sim \rangle\), where \(s \sim t\) means that there is \(g \in \text{Aut}(M)\) such that \(g(s) = t\). That is, we replace the named subsets of \(M\) by the single binary relation \(\sim\).

(c) A class \(L\) of U-trees is faithful, if for every \(M, N \in L\): if \(\text{Aut}(M) \cong \text{Aut}(N)\), then \(T(M) \cong T(N)\). Clearly, for trees: \(T(M) \cong T(N) \Rightarrow M \cong N\). So, for a faithful class \(L\) of trees and \(M, N \in L\): \(\text{Aut}(M) \cong \text{Aut}(N) \Rightarrow M \cong N\).

The answer to the main question cannot be easily stated. One ingredient in the answer, is finding large faithful classes of U-trees. Theorem 1 below is an example of such a result. Our more general theorems come close to showing that the faithful class of theorem 1 cannot be enlarged.

**Definition:** Let \(M = \langle T, <, \ldots \rangle\) be a U-tree. (a) \(\text{Max}(M)\) denotes the set of maximal elements of \(M\). \(A \subseteq T\) is an interval of \(M\), if \(A\) is linearly ordered by \(<\), and for every \(s, t \in A\) and \(u \in T\): if \(s < u < t\), then \(u \in A\). Let \(s, t \in T\). \(\text{Or}(t; s)\) denotes \(\{f(t) | f \in \text{Aut}(M) \text{ and } f(s) = s\}\). \(t\) is called a successor of \(s\) (\(t \in \text{Suc}(s)\)), if \(s < t\) and \(\{s, t\}\) is an interval.

(b) \(M\) is complete, if: (1) Every \(\emptyset \neq A \subseteq T\) has an infimum (\(\inf(A)\)); (2) Every interval in \(T\) has a supremum; and (3) If \(A\) and \(B\) are disjoint nonempty intervals, then \(\inf(A) \neq \inf(B)\).
Every U-tree \( M \) has a naturally defined completion \( N \). The named subsets of \( N \) are: \( M \), and all the named subsets of \( M \). It follows that \( Aut(N) \cong Aut(M) \). So, when seeking faithful classes, we may consider only complete U-trees.

**Theorem 1:** The class of all U-trees \( M \) satisfying conditions (1)-(4) below, is faithful. (1) \( M \) is complete. (2) For every \( s \in M \): \( |\text{Suc}(s)| \neq 1 \). (3) For every \( s \in M \): either for all \( u, v \in \text{Suc}(s) \): \( u \sim v \), or for all distinct \( u, v \in \text{Suc}(s) \): \( u \not\sim v \). (4) For every \( s \in M \) and \( t > s \): if \( |\text{Or}(t; s)| \leq 2 \), then \( t \in \text{Suc}(s) - \text{Max}(M) \).

For the class of \( \aleph_0 \)-categorical U-trees, the main question has the following complete answer. A class \( K_{\text{CAT}} \) of \( \aleph_0 \)-categorical U-trees is defined by listing five properties similar to properties (1)-(4) of theorem 1. (See 0.6.) We prove that \( K_{\text{CAT}} \) is faithful, and that if \( M \) is an \( \aleph_0 \)-categorical U-tree, then there is \( N \in K_{\text{CAT}} \) such that \( Aut(M) \cong Aut(N) \).

The situation in the class of all trees is similar, but more complex.
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Trees, sometimes called semilinear orders, are partially ordered sets in which every initial segment determined by an element is linearly ordered. This book focuses on automorphism groups of trees, providing a nearly complete analysis of when two trees have isomorphic automorphism groups. Special attention is paid to the class of $\aleph_0$-categorical trees, and for this class the analysis is complete. Various open problems, mostly in permutation group theory and in model theory, are discussed, and a number of research directions are indicated. Aimed at graduate students and researchers in model theory and permutation group theory, this self-contained book will bring readers to the forefront of research on this topic.