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# The Structure of Relation Algebras Generated by Relativizations 

Steven R. Givant



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# CONTEMPORARY Mathematics 

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Steven R. Givant



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To the memory of Alfred Tarski, my teacher and friend.

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## Introduction

The foundation of an algebraic theory of binary relations was laid by C. S. Peirce, building on earlier work of Boole and De Morgan (see Peirce [1882]). The basic universe of discourse of this theory is a collection of binary relations over some set, and the basic operations on these relations are those of forming unions, complements, compositions, and inverses. There is also a distinguished relation, the identity relation. Other operations and distinguished relations studied by Peirce are definable in terms of the ones just mentioned. Let us call such an algebra of relations a proper relation algebra.

A modern development of this theory as a theory of abstract algebras,

$$
\mathfrak{A}=\left\langle A,+,^{-}, ;,^{-}, 1^{\prime}\right\rangle
$$

axiomatized by a finite set of equations, was undertaken by Tarski and his students and colleagues, beginning around 1940. In this development, $\left\langle A,,^{-}\right\rangle$is an abstract Boolean algebra, the operations ; and ${ }^{〔}$ are abstract versions of relational composition and inversion, and 1 ' is a distinguished constant with properties similar to those of the identity relation.

Chin-Tarski [1951] contains an axiomatic study of the arithmetic of these abstract relation algebras. Special types of elements are defined-often, these are abstractions of well-known types of binary relations-and laws about such elements are derived from the basic postulates of the theory. To give some examples, equivalence elements $e$ and functional elements $f$ of a relation algebra $\mathfrak{A}$-abstractions of equivalence relations and functions-are defined respectively by the following conditions:

$$
e ; e \leq e \quad \text { and } \quad e^{\smile} \leq e \quad, \quad f^{\smile} ; f \leq 1
$$

It is shown, for example, that $f$ is a functional element iff it satisfies the following distributive law:

$$
f ;(a \cdot b)=(f ; a) \cdot(f ; b) \quad \text { for all } a, b \in A
$$

(here, "." denotes Boolean multiplication).
Building on op. cit. and on Jónsson-Tarski [1951], Jónsson-Tarski [1952] undertakes a study of the algebraic properties of relation algebras. For example, the notion of a relation algebraic ideal is introduced, and its connections with homomorphisms are established. Ideals are characterized in terms of special elements $e$, called ideal elements, that satisfy the equation $1 ; e ; 1=e$ (here, 1 is the Boolean unit). Ideal elements are particular examples of equivalence elements, and they can be alternately characterized as elements $e$ that satisfy the distributive law

$$
e \cdot(a ; b)=(e \cdot a) ;(e \cdot b) \quad \text { for all } a, b \in A
$$

With the help of this characterization, one shows that, for an arbitrary element $e$ of $\mathfrak{A}$, the mapping $a \longmapsto a \cdot e$ is a homomorphism on $\mathfrak{A}$ iff $e$ is an ideal element. In this case, the homomorphic image of $\mathfrak{A}$ under the mapping is the algebra

$$
\mathfrak{A}(e)=\left\langle A(e),+,^{-e}, ;,^{-}, 1^{\prime} \cdot e\right\rangle,
$$

where $A(e)=\{a \in A: a \leq e\}$, while ${ }^{-e}$ is complementation relative to $e$, and the other operations have the same meaning as in $\mathfrak{A}$. The algebra $\mathfrak{A}(e)$ is called the relativization of $\mathfrak{A}$ to $e$. One can actually form the relativization of a relation algebra to an arbitrary equivalence element, and not just to an ideal element.

Relativizations play an important role, not only in the study of homomorphic images of relation algebras, but also in the study of direct decompositions. For example, Jónsson and Tarski show that a relation algebra $\mathfrak{A}$ is isomorphic to a direct product $\mathfrak{B} \times \mathfrak{C}$ iff there is an ideal element $e$ in $\mathfrak{A}$ such that $\mathfrak{A}(e) \cong \mathfrak{B}$ and $\mathfrak{A}\left(e^{-}\right) \cong \mathfrak{C}$. Using this, they then prove that directly indecomposable-and hence also subdirectly indecomposable-relation algebras are simple. Hence, every relation algebra is the subdirect product of simple relation algebras.

One of the main focuses of Jónsson-Tarski [1952] is the study of representation problems: when is an abstract relation algebra representable as, i.e., isomorphic to, a proper relation algebra? Lyndon [1950] proves that not every abstract relation algebra is representable in this way. Thus, an important task of the algebraic theory is to find interesting and general conditions that do imply representability. For example, using the results of Jónsson-Tarski [1951], Jónsson and Tarski prove that every relation algebra is embeddable into a complete and atomic relation algebra. Further, they prove that an atomic relation algebra in which every atom is functional must be representable. The distributive law for functional elements that was mentioned before plays an important role in the proof.

In addition to their central role in the study of homomorphisms and direct decompositions, relativizations can also play an important role in establishing representation theorems and in describing relation algebras generated by certain kinds of elements. Imagine, for example, that we wish to describe the subalgebra of a relation algebra $\mathfrak{A}$ generated by a certain sequence of elements, $\left\langle a_{\gamma}: \gamma \in \Gamma\right\rangle$, in $\mathfrak{A}$. One might proceed as follows:
(i) Construct a suitable sequence $\left\langle e_{\xi}: \xi \in \Xi\right\rangle$ of pairwise disjoint equivalence elements that are generated by the sequence $\left\langle a_{\gamma}: \gamma \in \Gamma\right\rangle$, and such that each $a_{\gamma}$ is below some $e_{\xi}$.
(ii) Describe the subalgebra $\mathfrak{B}_{\xi}$ of $\mathfrak{A}\left(e_{\xi}\right)$ generated by $\left\{a_{\gamma}: a_{\gamma} \leq e_{\xi}\right\}$.
(iii) Describe the subalgebra of $\mathfrak{A}$ generated by $\left\langle\mathfrak{B}_{\xi}: \xi \in \Xi\right\rangle$, i.e., generated by the set $\bigcup_{\xi \in \Xi} B_{\xi}$.

For example, Jónsson [1988] describes all relation algebras generated by a single equivalence element $a$. In particular, he proves that all such algebras are representable. His proof can be seen to have the following form. First, he shows that $a$ generates three pairwise disjoint equivalence elements, $a_{1}, a_{2}$, and $a_{3}$, such that $a=a_{1}+a_{2}+a_{3}$ and

$$
\begin{equation*}
\left(a_{1} \cdot 0^{\prime}\right) ;\left(a_{1} \cdot 0^{\prime}\right)=0 \quad, \quad\left(a_{2} \cdot 0^{\prime}\right) ;\left(a_{2} \cdot 0^{\prime}\right)=a_{2} \cdot 1^{\prime} \quad, \quad\left(a_{3} \cdot 0^{\prime}\right) ;\left(a_{3} \cdot 0^{\prime}\right)=a_{3} \tag{1}
\end{equation*}
$$

(here, $0^{\prime}$ is the complement of $\left.1^{\prime}\right)$. Next, he defines $1_{i}=\left(a_{i} \cdot 1^{\prime}\right) ; 1 ;\left(a_{i} \cdot 1^{\prime}\right)$ for $i=1,2,3$, and he observes that $1_{1}, 1_{2}$, and $1_{3}$ are pairwise disjoint equivalence elements with $a_{i} \leq 1_{i}$ for each $i$. With the essential help of (1), he describes the subalgebra $\mathfrak{B}_{i}$ of $\mathfrak{A}\left(1_{i}\right)$ generated by $a_{i}$, under the assumption that $\mathfrak{A}$ is simple. (This is not an essential restriction, in view of the subdirect decomposition theorem referred to above.) In particular, he shows that each $\mathfrak{B}_{i}$ is finite and representable. Finally, still assuming that $\mathfrak{A}$ is simple, he describes the subalgebra of $\mathfrak{A}$ generated by $B_{1} \cup B_{2} \cup B_{3}$. He proves, in the process, that it must be finite and representable. Orally, Jónsson posed the problem whether a relation algebra generated by a finite chain of equivalence elements is necessarily finite and representable.

The following questions emerge from our discussion.
(I) Let $\left\langle e_{\xi}: \xi \in \Xi\right\rangle$ be a sequence of pairwise disjoint equivalence elements in a relation algebra $\mathfrak{A}$, and, for each $\xi \in \Xi$, suppose that $\mathfrak{B}_{\xi}$ is a subalgebra of the relativization $\mathfrak{A}\left(e_{\xi}\right)$. Can we describe the subalgebra, $\mathfrak{S g}^{\mathfrak{A}}\left(\bigcup_{\xi \in \Xi} B_{\xi}\right)$, generated by $\left\langle\mathfrak{B}_{\xi}: \xi \in \Xi\right\rangle$ in $\mathfrak{A}$ ?
(II) What properties of the algebras $\mathfrak{B}_{\xi}$ does $\mathfrak{S g}^{\mathfrak{A}}\left(\bigcup_{\xi \in \Xi} B_{\xi}\right)$ inherit? For example, if each $\mathfrak{B}_{\xi}$ is representable, must $\mathfrak{S g}^{\mathfrak{2}}\left(\bigcup_{\xi \in \Xi} B_{\xi}\right)$ be representable? If $\Xi$ is finite, and if each $\mathfrak{B}_{\xi}$ is finite, must $\mathfrak{S g}^{\mathfrak{A}}\left(\bigcup_{\xi \in \Xi} B_{\xi}\right)$ be finite?

The main purpose of the present work is to give a complete description of $\mathfrak{S g}^{\mathfrak{A}}\left(\bigcup_{\xi \in \Xi} B_{\xi}\right)$ in terms of the elements and operations of the algebras $\mathfrak{B}_{\xi}$. As a consequence of this description, we will obtain, for example, affirmative answers to the two last questions posed in (II). Moreover, in the case when $\mathfrak{A}$ is simple, we shall show that certain other properties, such as atomicity, are also inherited by $\mathfrak{S g}^{\mathfrak{A}}\left(\bigcup_{\xi \in \Xi} B_{\xi}\right)$.

To give a flavor of the description, let's concentrate on the case when $\mathfrak{A}$ is simple and there is just one equivalence element, $e$, in our sequence. Moreover, let's assume that $e$ is reflexive, i.e., $1^{\prime} \leq e$. Given a subalgebra $\mathfrak{B}$ of $\mathfrak{A}(e)$, we want to describe $\mathfrak{S g}^{\mathfrak{A}} B$. We shall prove that the elements of $\mathfrak{S g}^{\mathfrak{A}} B$ are just the finite sums of elements of $\mathfrak{B}$ and elements of the form $x ; e^{-} ; y$, where $x$ and $y$ are subidentity elements of $\mathfrak{B}$, i.e., elements of $\mathfrak{B}$ that are below the identity. For reasons that will become clear later, we call elements of the form $x ; e^{-} ; y$ rectangles. They are disjoint from the elements of $\mathfrak{B}$.

Most of the operations of $\mathfrak{S g}^{2} B$ can be described rather easily in terms of the operations of $\mathfrak{B}$. For example, if $x, y$ are subidentity elements, and $a$ an arbitrary element, of $\mathfrak{B}$, then

$$
\begin{aligned}
\left(x ; e^{-} ; y\right)^{-} & =x^{-1^{\prime}} ; e^{-} ; y+x ; e^{-} ; y^{-1^{\prime}}+x^{-1^{\prime}} ; e^{-} ; y^{-1^{\prime}}+e \\
\left(x ; e^{-} ; y\right)^{-} & =y ; e^{-} ; x, \\
\left(x ; e^{-} ; y\right) \cdot\left(u ; e^{-} ; v\right) & =(x \cdot u) ; e^{-} ;(y \cdot v) \\
a ;\left(x ; e^{-} ; y\right) & =\left(a ; x ; e \cdot 1^{\prime}\right) ; e^{-} ; y
\end{aligned}
$$

(in the first equation, $x^{-1}$ denotes the complement of $x$ relative to $1^{\prime}$ ). The description of the relative product of two rectangles, $x ; e^{-} ; y$ and $u ; e^{-} ; v$, however, is more delicate. It has one of four possible values, depending on the "size" of the ideal element $e ;(y \cdot v) ; e$ of $\mathfrak{B}$. To handle this situation, we introduce a measure $\|\|$ on the ideal elements of $\mathfrak{B}$. The measure has one of four possible values, 0 ,

1,2 , or 3 (all numbers greater than or equal to 3 are identified with $\infty$ ), according as the ideal element is either 0 or "spans" 1,2 , or at least 3 of the "equivalence classes" of $e$. Set $b=e ;(y \cdot u) ; e$. We then have:

$$
\left(x ; e^{-} ; y\right) ;\left(u ; e^{-} ; v\right)=\left\{\begin{array}{ll}
0 & \text { if }\|b\|=0 \\
\left(x \cdot b^{-1^{1}}\right) ; 1 ;\left(v \cdot b^{-1^{\prime}}\right) & \text { if }\|b\|=1 \\
\left(x \cdot b^{-1^{1}}\right) ; 1 ; v+x ; 1 ;\left(v \cdot b^{-1^{\prime}}\right)+x ; e ; v & \text { if }\|b\|=2 \\
x ; 1 ; v & \text { if }\|b\|=3
\end{array} .\right.
$$

This completes the description of $\mathfrak{S g}^{\mathfrak{A}} B$. Using it, we shall readily show that if $\mathfrak{B}$ is finite, atomic, integral, or finitely decomposable (into a direct product of simple algebras), then so is $\mathfrak{S g}^{\mathfrak{a}} B$.

It is natural to ask: Upon what essential properties of $\mathfrak{B}$ and $\mathfrak{A}$ does the structure of $\mathfrak{S g}^{\mathfrak{a}} B$ depend? Certainly, the isomorphism type of $\mathfrak{B}$ and the value of \|\| \| on ideal elements of $\mathfrak{B}$ are important. As it turns out, nothing else plays an essential role. For a precise statement of this result, let $\mathfrak{A}$ and $\mathfrak{A}^{\prime}$ be simple relation algebras, $e$ and $e^{\prime}$ reflexive equivalence elements in $\mathfrak{A}$ and $\mathfrak{A}^{\prime}$, and $\mathfrak{B}$ and $\mathfrak{B}^{\prime}$ subalgebras of $\mathfrak{A}(e)$ and $\mathfrak{A}^{\prime}\left(e^{\prime}\right)$ respectively. Suppose that $\vartheta$ is an isomorphism of $\mathfrak{B}$ onto $\mathfrak{B}^{\prime}$ which preserves measure, i.e.,

$$
\|b\|=\|\vartheta(b)\| \quad \text { for every ideal element } b \text { of } \mathfrak{B}
$$

Then, as we will prove, $\vartheta$ can be (uniquely) extended to an isomorphism of $\mathfrak{S g}^{\mathfrak{2}} B$ onto $\mathfrak{S g}^{\mathfrak{2 d}^{\prime}} B^{\prime}$.

We turn now to another, though ultimately related, problem: Does every relation algebra $\mathfrak{B}$ "sit inside of" some simple relation algebra? If so, is there a minimal simple relation algebra "containing" $\mathfrak{B}$, a kind of simple closure of $\mathfrak{B}$ ? If so, is this simple closure in some sense unique?

The unit of a relation algebra is always an equivalence element. Therefore, it is natural to interpret the first question as asking whether there exist a simple relation algebra $\mathfrak{A}$ and an equivalence element $e$ in $\mathfrak{A}$ such that $\mathfrak{B}=\mathfrak{A}(e)$. For $\mathfrak{A}$ to be minimal, it should "fit" around $\mathfrak{B}$ as "tightly" as possible. This means that $\mathfrak{A}$ and $\mathfrak{B}$ should have the same identity element, and that $\mathfrak{B}$ should generate $\mathfrak{A}$. Thus, we arrive at the following definition: a relation algebra $\mathfrak{A}$ is a simple closure of $\mathfrak{B}$ iff $\mathfrak{A}$ is simple, there is a reflexive equivalence element $e$ in $\mathfrak{A}$ such that $\mathfrak{B}=\mathfrak{A}(e)$, and $\mathfrak{A}=\mathfrak{S g}^{\mathfrak{A}} B$. Our problem can now be formulated quite precisely: Does every relation algebra have a simple closure, and, if so, in what sense is it unique?

We shall prove the following existence and uniqueness theorems for simple closures. Every relation algebra $\mathfrak{B}$ has a simple closure, and, in fact, for any appropriate four-valued measure $\mu$ on the ideal elements of $\mathfrak{B}$, there is a simple closure of $\mathfrak{B}$ such that $\|\|=\mu$. If $\mathfrak{B}$ is finite or representable, then so is each such simple closure. Moreover, if $\mathfrak{A}$ and $\mathfrak{A}^{\prime}$ are two simple closures of $\mathfrak{B}$, say $\mathfrak{B}=\mathfrak{A}(e)=\mathfrak{A}^{\prime}\left(e^{\prime}\right)$, and if the measure \|\| gives the same value in $\mathfrak{A}$ as it does in $\mathfrak{A}^{\prime}$ on the ideal elements of $\mathfrak{B}$, then $\mathfrak{A}$ is isomorphic to $\mathfrak{A}^{\prime}$ via an isomorphism that is the identity mapping on $B$ and that preserves measure. In other words, the simple closure is uniquely determined by the isomorphism type of $\mathfrak{B}$ and the measure $\mu$. This uniqueness is a consequence of the isomorphism theorem that we described above. As a result of the existence and uniqueness theorems, we can show that, e.g., a relation algebra with $2^{n}$ ideal elements has, up to isomorphisms, exactly $3^{n}$
simple closures. As a further consequence, we show that if $\mathfrak{B}$ is a subalgebra of a relativization $\mathfrak{A}(e)$, then $\mathfrak{S g}^{\mathfrak{A}} B$ is representable iff $\mathfrak{B}$ is representable.

We shall also prove a generalization of the existence and uniqueness theorems: every sequence $\left\langle\mathfrak{B}_{\xi}: \xi \in \Xi\right\rangle$ of essentially disjoint (except for 0 ) relation algebras, with a corresponding sequence of appropriate measures $\left\langle\mu_{\xi}: \xi \in \Xi\right\rangle$, has a simple closure such that \|\| agrees with $\mu_{\xi}$ on the ideal elements of $\mathfrak{B}_{\xi}$ for each $\xi$. Again, the simple closure is uniquely determined by the isomorphism types of the algebras in the sequence and by the measures. It is representable iff each of the algebras $\mathfrak{B}_{\xi}$ is representable. We then establish the representation theorem mentioned in (II): $\mathfrak{S g}^{\mathfrak{A}}\left(\bigcup_{\xi \in \Xi} B_{\xi}\right)$ is representable iff each $\mathfrak{B}_{\xi}$ is representable.

As an application of our results, we investigate relation algebras generated by systems of equivalence elements. Let $\mathfrak{T}=\langle T, \preccurlyeq\rangle$ be a tree. We define a system $\left\langle e_{t}: t \in T\right\rangle$ of equivalence elements in some relation algebra to be a tree of equivalence elements (indexed by $\mathfrak{T}$ ) iff $e_{s} \geq e_{t}$ whenever $s \preccurlyeq t$, and $e_{s} \cdot e_{t}=0$ whenever $s$ and $t$ are incomparable in $\mathfrak{T}$. We shall prove that any relation algebra $\mathfrak{A}$ generated by a tree of equivalence elements (or even by a pseudo-tree of equivalence elements) is representable. Moreover, if the tree is finite, then so is $\mathfrak{A}$. As a consequence, we see that Jónsson's problem has an affirmative solution: every relation algebra generated by a finite chain of equivalence elements is finite and representable. We give examples to show that, in some sense, our results are the best possible. For example, it is not true that an arbitrary finite sequence of equivalence elements generates a relation algebra that is either finite or representable.

There are certain philosophical implications of the existence theorems for simple closures that should perhaps be mentioned. We usually think of the simple algebras-in the case of relation algebras these coincide with the subdirectly irreducible algebras-as the most basic building blocks by means of which the more complicated algebras of a theory can be constructed. In some sense, the simple algebras seem to be the easiest algebras to understand and to work with. The existence theorem shows that, as far as relation algebras are concerned, this belief is illusory. Simple algebras contain, within their algebraic structure, all of the complexity of arbitrary relation algebras, because they contain arbitrary relation algebras as relativizations. Succinctly put: Simple algebras aren't simple!

We give, now, a brief outline of the organization of the book. Chapter 1 presents the basic definitions and laws that will be used throughout this work. The laws in $1.6,1.7,1.9,1.14$, and 1.16 are referred to quite frequently. Chapter 2 contains the basic algebraic material on homomorphisms, ideals, relativizations, direct and subdirect decompositions, and perfect extensions. The presentation of direct (and subdirect) decompositions is in terms of inner direct products-well known from group theory-as opposed to the usual outer, or Cartesian, direct products. Also, a notion of the inner direct sum of relation algebras is introduced. Theorem 2.18 contains a characterization of each type of inner decomposition. Some of the results in the chapter appear to be new, for example, 2.21, 2.25, and 2.26 .

Jónsson [1988] shows that if $e$ is an equivalence element in a simple relation algebra, then $e^{-} ; e^{-}$has one of exactly three values: $0, e$, or 1 . Unfortunately, for non-reflexive equivalence elements, we always have $e^{-} ; e^{-}=1$. It is more useful to look at the value of $e^{-} ; e^{-}$relative to $e ; 1 ; e$. In 3.5 we prove that, in a simple relation algebra, either $e$ is 0 , or else it is different from 0 and $\left[e^{-} \cdot(e ; 1 ; e)\right] ;\left[e^{-} \cdot(e ; 1 ; e)\right]$ has exactly one of the three values $0, e$, or $e ; 1 ; e$. Thus, we can assign a unique
number, $0,1,2$, or 3 , to $e$ according to which of these four cases holds. We denote this number by $\|e\|$, and call it the characteristic of $e$. If $e$ is a symmetric, transitive relation on a non-empty set $U$, then $\|e\|$ is $0,1,2$, or 3 in the full set relation algebra on $U$ iff $e$ has zero, one, two, or at least three equivalence classes (as an equivalence relation on its field). In an arbitrary relation algebra $\mathfrak{A}$, not every equivalence element $e$ has a characteristic. We show, however, in Theorem 3.12 that $\mathfrak{A}$ and $e$ can be decomposed (or "factored") into pieces $\mathfrak{A}_{i}$ and $e_{i}$, for $i=0,1,2,3$, so that $\left\|e_{i}\right\|=i$ in $\mathfrak{A}$ and in $\mathfrak{A}_{i}$.

Since an ideal element is always an equivalence element, every ideal element $b$ of $\mathfrak{A}(e)$ will have a characteristic, $\|b\|$, when $\mathfrak{A}$ is simple. In set-theoretic terms, $\|b\|$ is a measure of the number of equivalence classes of $e$ that $b$ spans. Thus, the notion of the characteristic of an ideal element of $\mathfrak{A}(e)$ leads to the concept of a positive measure on these ideal elements. It is natural to restrict the range of such measures to four values, $0,1,2$, and 3 , because relation algebras can't count beyond 3 , i.e., they can't distinguish the numbers greater than or equal to 3 from one another, or from $\infty$. In 3.16 we prove that, in a simple relation algebra $\mathfrak{A},\| \|$ is a completely additive measure on the Boolean algebra of ideal elements of $\mathfrak{A}(e)$. Finally, we show in 3.26 that, if an isomorphism $\vartheta$ between relativizations $\mathfrak{A}(e)$ and $\mathfrak{A}^{\prime}\left(e^{\prime}\right)$ preserves $\|\|$, then the canonical extension of $\vartheta$ to the perfect extensions of $\mathfrak{A}(e)$ and $\mathfrak{A}^{\prime}\left(e^{\prime}\right)$ also preserves \|\|.

Chapter 4 develops the arithmetic of rectangles necessary to describe the operations of $\mathfrak{S g}^{\mathfrak{A}} B$ when $\mathfrak{B} \subseteq \mathfrak{A}(e)$. The most involved of these laws are the ones describing the relative product of two rectangles. They are given in 4.7.

The principal results of the book are contained in Chapters 5-7. Theorem 5.1 gives the description of the subalgebra, $\mathfrak{S g}^{\mathfrak{A}} B$, generated in $\mathfrak{A}$ by a subalgebra $\mathfrak{B}$ of $\mathfrak{A}(e)$. The simplification of 5.1 that results when we assume $\mathfrak{A}$ to be simple is given in 5.3 . We conclude in 5.7 that if $\mathfrak{B}$ is, e.g., integral or finitely decomposable, then so is $\mathfrak{S g}^{\mathfrak{A}} B$. When $\mathfrak{A}$ is simple, $\mathfrak{S g}^{\mathfrak{A}} B$ also inherits atomicity from $\mathfrak{B}$. The generalizations of 5.1 and 5.3 to finite sequences $\left\langle e_{\xi}: \xi \in \Xi\right\rangle$ of pairwise disjoint equivalence elements of $\mathfrak{A}$, and $\left\langle\mathfrak{B}_{\xi}: \xi \in \Xi\right\rangle$, with $\mathfrak{B}_{\xi}$ a subalgbera of $\mathfrak{A}\left(e_{\xi}\right)$ for each $\xi$, are given in 5.10 and 5.11 , and the generalizations to infinite sequences are given in 5.16 and 5.17 . In the case when $\mathfrak{A}$ is not necessarily simple, but at any rate $\mathfrak{B}$ has only finitely many ideal elements, we give-in 5.21 -an explicit decomposition of $\mathfrak{S g}^{\mathfrak{2}} B$ into finitely many simple factors.

Chapter 6 begins with the isomorphism theorems, 6.1 and 6.2 , for $\mathfrak{S g}^{\mathfrak{A}} B$ and $\mathfrak{S g}^{\mathfrak{A}}\left(\bigcup_{\xi \in \Xi} B_{\xi}\right)$, that we referred to earlier. As a direct consequence of these, we obtain the uniqueness theorem for simple closures, 6.5. Theorems 6.6 and 6.10 contain the existence theorems for simple closures of representable relation algebras and abstract relation algebras, respectively. The generalizations to arbitrary sequences of (almost disjoint) representable relation algebras and abstract relation algebras are given in 6.7 and 6.11 respectively. As a consequence of $6.2,6.6$, and 6.7, we prove in 6.8 and 6.9 that $\mathfrak{S g}^{\mathfrak{A}} B$ is representable iff $\mathfrak{B}$ is representable, and that $\mathfrak{S g}^{\mathfrak{A}}\left(\bigcup_{\xi \in \Xi} B_{\xi}\right)$ is representable iff each $\mathfrak{B}_{\xi}$ is representable.

The representation theorems for relation algebras generated by trees or pseudotrees of equivalence elements are given in 7.2 and 7.3. The chapter concludes with a sequence of counterexamples to show that various suggested extensions of our results do not hold.

I have tried to make the results of the book accessible to readers who are not
experts in the theory of relation algebras. Readers familiar with the field might want to skip over the material in Chapters 1 and 2 initially. They could begin with Chapter 3, then skip to the main results in Chapters 5, 6, and 7, referring back to the earlier material, in particular, the laws in Chapter 4, as needed.

It is a pleasure to acknowledge my indebtedness to those who helped me during the course of this work. First of all, it was Bjarni Jónsson who posed the original problem (mentioned above) that eventually led to the results presented here. Without his interest in the subject, this monograph would never have been written. I am also very grateful to Hajnal Andréka, Peter Jipsen, Roger Maddux, István Németi, and Richard Thompson for their interest in this research, for their stimulating discussions, and for their individual contributions-which are, of course, properly credited to them at appropriate places. The research, itself, was supported in part by the Letts-Villard endowed professorship at Mills College, and in part by a grant from the Institute for Research and Exchanges, IREX. Some of the results were announced at the Icelandic Symposium in honor of Bjarni Jónsson, 1990, at a conference on algebraic logic held at Mills College, 1990, and in Givant [1990]. Some extensions of the results in this book to the theory of cylindric algebras have been obtained by the author in collaboration with Hajnal Andréka and István Németi. These results will appear elsewhere.

## The Structure of Relation Algebras Generated by Relativizations

## Steven R. Givant

The foundation for an algebraic theory of binary relations was laid by De Morgan, Peirce, and Schröder during the second half of the nineteenth century. Modern development of the subject as a theory of abstract algebras, called "relation algebras", was undertaken by Tarski and his students. This book aims to analyze the structure of relation algebras that are generated by relativized subalgebras. As examples of their potential for applications, the main results are used to establish representation theorems for classes of relation algebras, and to prove existence and uniqueness theorems for simple closures (i.e., for minimal simple algebras containing a given family of relation algebras as relativized subalgebras). The book is well-written and accessible to those who are not specialists in this area. In particular, it contains two introductory chapters on the arithmetic and the algebraic theory of relation algebras. It would be suitable for use in graduate courses on algebras of binary relations or algebraic logic.

