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Laver's Forcing and Outer Measure

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ABSTRACT. We show that preservation of outer measure by Laver's forcing can be in a natural way obtained from Sierpiński's inductive analysis of analytic sets.

1. Let $\mathbb{L}$ be Laver's forcing (see [**L**]) and let μ be the Lebesgue measure in ${}^{\omega}2$. We shall prove the following theorem of Woodin [**W**] , rediscovered later by Judah and Shelah [**JS**] ([**W**] is unpublished and the proof in [**JS**] is somewhat difficult to follow).

THEOREM. *For* $A \subseteq {}^{\omega}2$, $\mu^*(A) = a \Rightarrow \mathbb{L} \Vdash \mu^*(A) = a$.

2. We first fix some notation.

DEFINITION. Write Ω for ${}^{<\omega}\omega$. Let $T \in \mathbb{L}$. For $t \in T$ let $T_t = \{s \in T : s \subseteq t \vee t \subseteq s\}$. For $\tau \in \Omega$ let $T(\tau)$ be the image of τ under the canonical isomorphism of Ω and T and let $T\langle\tau\rangle = T_{T(\tau)}$ (so, $T(\emptyset)$ is the stem of T).

For $S, T \in \mathbb{L}$ and $n \in \omega$ write $S \leq_n T$ iff $\forall \tau \in {}^{n}\omega\ S(\tau) = T(\tau)$. Note that if $T_{n+1} \leq_n T_n$ $(n \in \omega)$ then $T = \bigcap_n T_n$ belongs to $\mathbb{L}$ and $T \leq_n T_n$.

For open $D \subseteq \mathbb{L}$ let

$$\widetilde{D} = \{T \in \mathbb{L} : \forall S \leq_0 T\ S \notin D\} \cup D,$$
$$D^* = \{T \in \mathbb{L} : \exists n\ \forall \tau \in \Omega\ |\tau| \geq n \Rightarrow T\langle\tau\rangle \in \widetilde{D}\}.$$

Note that D^* is open and $D \subseteq D^*$.

LEMMA. *If* $D \subseteq \mathbb{L}$ *is open and nonempty below* $S \leq T \in D^*$ *then* $\exists s \in S\ T_s \in D$.

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PROOF. Let $R \in D$, $R \leq S$. Choose $s \in R$ long enough to guarantee $R_s \leq_0 T_s \in \widetilde{D}$. Since D is open, $R_s \in D$, hence by $R_s \leq_0 T_s \in \widetilde{D}$ we have $T_s \in D$. □

3. The following lemma is implicit in [**L**].

LEMMA.
1. *Suppose $D \subseteq \mathbb{L}$ is open. Then $\forall i\ \forall T\ \exists S \leq_i T\ S \in D^*$.*
2. *Suppose $D_n \subseteq \mathbb{L}$ ($n \in \omega$) are open. Then $\forall i\ \forall T\ \exists S \leq_i T\ S \in \bigcap_n D_n^*$.*

PROOF. (1) Let $S^{-1} = T$ and for $k \in \omega$ let $S^k = \bigcup_{\sigma \in {}^{i+k}\omega} R^\sigma$, where R^σ's are chosen so that $R^\sigma \leq_0 S^{k-1}\langle\sigma\rangle$ and $R^\sigma \in D$ whenever possible. Note that $S^k \leq_{i+k} S^{k-1}$. Let $S = \bigcap_k S^k$. Then $S \leq_i T$ and $\forall k \geq 0\ \forall \sigma \in {}^{i+k}\omega\ S\langle\sigma\rangle \in \widetilde{D}$.

(2) By (1) find $\langle S_n : n \in \omega \rangle$ such that $S_n \in D_n^*$ and

$$\cdots \leq_{i+n+1} S_n \leq_{i+n} \cdots \leq_{i+1} S_0 \leq_i T.$$

Let $S = \bigcap_n S_n$. □

4. The next lemma is motivated by Sierpiński's proof that every analytic set is a union and intersection of $\aleph_1$ Borel sets (see [**K**]).

LEMMA. *Let $A_\sigma \subseteq {}^\omega 2$ ($\sigma \in \Omega$) be such that $\forall \sigma\ \mu^*(A_\sigma) \leq a$ and $\forall \sigma\ A_\sigma \subseteq \liminf_n A_{\sigma^\frown n}$. Then*

$$\mu^*\Big(\bigcap_{\Sigma \leq_0 \Omega} \bigcup_{\sigma \in \Sigma} A_\sigma\Big) \leq a.$$

PROOF. Without loss of generality A_σ's are Borel. Define A_σ^α ($\alpha < \omega_1, \sigma \in \Omega$) as follows

$$\begin{aligned} A_\sigma^0 &= A_\sigma, \\ A_\sigma^{\alpha+1} &= \liminf_n A_{\sigma^\frown n}^\alpha, \\ A_\sigma^\lambda &= \bigcup_{\alpha<\lambda} A_\sigma^\alpha, \text{ for limit } \lambda. \end{aligned}$$

Note that for a fixed σ, $A_\sigma^\alpha \subseteq \liminf_n A_{\sigma^\frown n}^\alpha$, the sets A_σ^α increase with α and have measure $\leq a$. So, there exist $\alpha_\sigma < \omega_1$ such that $\forall \beta \geq \alpha_\sigma\ \mu(A_\sigma^{\beta+1} \setminus A_\sigma^\beta) = 0$. Let $\alpha = \sup_\sigma \alpha_\sigma$. Then $\mu(A_\emptyset^{\alpha+1} \cup \bigcup_\sigma (A_\sigma^{\alpha+1} \setminus A_\sigma^\alpha)) \leq a$. We are done by the following claim.

CLAIM.

$$\bigcap_{\Sigma \leq_0 \Omega} \bigcup_{\sigma \in \Sigma} A_\sigma \subseteq A_\emptyset^{\alpha+1} \cup \bigcup_\sigma (A_\sigma^{\alpha+1} \setminus A_\sigma^\alpha).$$

PROOF. Let $x \notin A_\emptyset^{\alpha+1} \cup \bigcup_\sigma (A_\sigma^{\alpha+1} \setminus A_\sigma^\alpha)$. Then, by $x \notin A_\emptyset^{\alpha+1}$, $\exists^\infty n\ x \notin A_{\langle n\rangle}^\alpha$. Together with $x \notin \bigcup_\sigma (A_\sigma^{\alpha+1} \setminus A_\sigma^\alpha)$ this gives $\exists^\infty n\ x \notin A_{\langle n\rangle}^{\alpha+1}$. By a similar argument, if $x \notin A_{\langle n\rangle}^{\alpha+1}$ then $\exists^\infty m\ x \notin A_{\langle n,m\rangle}^{\alpha+1}$. Thus $\exists^\infty n\ \exists^\infty m\ x \notin A_{\langle n,m\rangle}^{\alpha+1}$. In this way we construct $\Sigma \leq_0 \Omega$ such that $x \notin \bigcup_{\sigma \in \Sigma} A_\sigma^{\alpha+1}$. It follows by $A_\sigma \subseteq A_\sigma^{\alpha+1}$ that $x \notin \bigcup_{\sigma \in \Sigma} A_\sigma$. □

5. We need the following definition.

DEFINITION. Suppose that $\mathbb{P}$ is a poset and c is a $\mathbb{P}$ name for a clopen subset of ${}^\omega 2$. Let

$$\operatorname{dom}(c) = \{p \in \mathbb{P} : p \text{ decides the value of } c\}.$$

For $p \in \operatorname{dom}(c)$ let $c(p)$ be the clopen that p chooses for c. Let

$$\mu(c) = \sup\{\mu(c(p)) : p \in \operatorname{dom}(c)\}.$$

Note that $\operatorname{dom}(c)$ is an open dense subset of $\mathbb{P}$ and if $p \leq q$ are in $\operatorname{dom}(c)$ then $c(p) = c(q)$.

LEMMA. *Let c be an $\mathbb{L}$ name for a clopen set. If $R \leq T$, $R \in \operatorname{dom}(c)$ and $T \in \operatorname{dom}(c)^*$, then $\exists s \in R\ c(T_s) = c(R)$.*

PROOF. Since $\operatorname{dom}(c)$ is nonempty below R, by Lemma 2, for some $s \in R$ we have $T_s \in \operatorname{dom}(c)$. Then $c(R) = c(R_s) = c(T_s)$. □

6. Now comes the basic estimation.

LEMMA. *Let c_n $(n \in \omega)$ be $\mathbb{L}$ names for clopen sets. If $T \in \bigcap_n \operatorname{dom}(c_n)^*$ then*

$$\mu^*(\{x \in {}^\omega 2 : T \Vdash x \in \bigcup_n c_n\}) \leq \sum_n \mu(c_n).$$

PROOF.

$$\begin{aligned} T \Vdash x \in \bigcup_n c_n &\Leftrightarrow \forall S \leq T\ \exists n\ \exists R \leq S\ x \in c_n(R) \\ &\Leftrightarrow \forall S \leq T\ \exists n\ \exists s \in S\ x \in c_n(T_s) \\ &\Leftrightarrow x \in \bigcap_{S \leq T} \bigcup_{s \in S} \bigcup_n c_n(T_s). \end{aligned}$$

(The second $\Rightarrow$ is by Lemma 5.) We conclude by Lemma 4. □

7. We are almost ready for the proof of Theorem 1.

DEFINITION. For a poset $\mathbb{P}$ and a real $\epsilon > 0$ let $\mathcal{I}^\epsilon_{\mathbb{P}}$ be the set of all sequences $c = \langle c_n : n \in \omega \rangle$ such that each c_n is a $\mathbb{P}$ name for a clopen of measure $\leq \epsilon/2^{n+1}$.

PROOF OF THEOREM 1. Suppose that $\mathbb{L} \not\Vdash \mu^*(A) = a$. Then there exist $T \in \mathbb{L}$, $\epsilon < a$ and $c \in \mathcal{I}^\epsilon_{\mathbb{L}}$ such that $T \Vdash A \subseteq \bigcup_n c_n$. By Lemma 3, we can assume that $T \in \bigcap_n \operatorname{dom}(c_n)^*$. Since $A \subseteq \{x \in {}^\omega 2 : T \Vdash x \in \bigcup_n c_n\}$, Lemma 6 yields $\mu^*(A) \leq \epsilon$, which is a contradiction. □

8. Unfortunately I am not able to prove that if the iterands of a countable support iteration of proper posets preserve outer measure (i.e. behave as $\mathbb{L}$ in Theorem 1) then so does the limit. So, let us introduce a property ★, which implies preservation of outer measure, and which (by a general preservation theorem of Shelah [**S**]) is itself preserved by countable support iterations of proper posets.

As usual let ZFC^* stand for a sufficiently large part of ZFC. If $M \models ZFC^*$, let $\operatorname{Ra}(M)$ be the set of reals which are random over M.

DEFINITION. Say that $\mathbb{P} \models \bigstar$ if, given a countable $N \prec H_\lambda$ such that $\mathbb{P} \in N$, given a rational $\epsilon > 0$, $c \in \mathcal{I}^\epsilon_\mathbb{P} \cap N$ and $p \in \mathbb{P} \cap N$, for any

$$x \in \mathrm{Ra}(N) \setminus \bigcap\{\bigcup_n c_n(p_n) : \langle p_n : n \in \omega\rangle \in (\prod_n \mathrm{dom}(c_n)) \cap N \text{ is a descending sequence below } p\},$$

there exists (N, P)-generic $q \leq p$ such that $q \Vdash x \in \mathrm{Ra}(N[G]) \setminus \bigcup_n c_n$.

LEMMA. *Suppose that* $\mathbb{P} \models \bigstar$. *Then, for* $A \subseteq {}^\omega 2$, $\mu^*(A) = a \Rightarrow \mathbb{P} \Vdash \mu^*(A) = a$.

PROOF. Suppose that $\mathbb{P} \models \bigstar$, $A \subseteq {}^\omega 2$, $\mu^*(A) = a$ and $\mathbb{P} \not\Vdash \mu^*(A) = a$. Then there exist $p \in \mathbb{P}$, a rational $\epsilon < a$ and a sequence $c \in \mathcal{I}^\epsilon_\mathbb{P}$ such that $p \Vdash A \subseteq \bigcup_n c_n$. Find a countable model $N \prec H_\lambda$ such that $\mathbb{P}$, p and c are in N. Find below p a descending sequence $\langle p_n : n \in \omega\rangle \in (\prod_n \mathrm{dom}(c_n)) \cap N$. Now, for $x \in \mathrm{Ra}(N) \setminus \bigcup_n c_n(p_n)$ we have by $\bigstar$, $p \not\Vdash x \in \bigcup_n c_n$. Hence $p \not\Vdash x \in A$ and thus $x \notin A$. It follows that $\mathrm{Ra}(N) \cap A \subseteq \bigcup_n c_n(p_n)$, hence $\mu^*(A) \leq \sum_n \mu(c_n(p_n)) \leq \epsilon$, which is a contradiction. □

9. We shall show the following theorem.

THEOREM. $\mathbb{L} \models \bigstar$.

For the proof we need two additional lemmas.

10. First we recall a Solovay-folklore lemma. We shall identify ${}^\omega 2$ and $\mathcal{P}(\omega)$.

LEMMA. *Suppose that N is a countable transitive model of ZFC^*, $p_0 \in N$, $\phi(\cdot,\cdot)$ is a Δ_0 formula with parameters $p_1, \ldots, p_n \in N$. Then*

1. *$\{x \subseteq \omega : \exists y \subseteq p_0\ \phi(x,y)\}$ is a Solovay set of reals over N (in the terminology of* [**Je**]*), i.e., there exists a formula ϕ^* with parameters in N such that for any $x \subseteq \omega$ with $N[x] \models ZFC^*$,*

$$\exists y \subseteq p_0\ \phi(x,y) \Leftrightarrow N[x] \models \phi^*(x);$$

2. *there exists a Borel set B coded in N such that for any $x \in \mathrm{Ra}(N)$,*

$$x \in B \Leftrightarrow \exists y \subseteq p_0\ \phi(x,y).$$

PROOF. (1) Let $W \in N$ be a transitive set such that $p_0, p_1, \ldots, p_n, \omega \in W$. There is a formula ψ such that for $x \subseteq \omega$,

$$\exists y \subseteq p_0\ \phi(x,y) \Leftrightarrow \exists y \subseteq W\ \langle W, x, y, \in\restriction W\rangle \models \psi(x, y, p_0, \ldots, p_n).$$

(Here x and y work as predicates; contexts like '$x \in z$' are replaced by '$\exists v \in z\ \forall w(x(w) \Leftrightarrow w \in v$'.) Let $\mathcal{C}$ be the poset adding a 1–1 function from W onto ω (i.e., $\mathcal{C}$ is the set of 1–1 functions from finite subsets of W into ω ordered by the reversed inclusion).

Fix $x \subseteq \omega$ such that $N[x] \models ZFC^*$ and let $c : W \to \omega$ be $\mathcal{C}$ generic function over $N[x]$. Let (†) stand for

$$\exists y \subseteq W\ \langle W, x, y, \in\restriction W\rangle \models \psi(x, y, p_0, \ldots, p_n)$$

and (‡) for

$$\exists y \subseteq \omega\ \langle \omega, c[x], y, c[\in\restriction W]\rangle \models \psi(c[x], y, c(p_0), \ldots, c(p_n)).$$

Then

$$\begin{aligned}(\dagger) &\Leftrightarrow (\ddagger)\\ &\Leftrightarrow N[x][c] \models (\ddagger)\\ &\Leftrightarrow N[x][c] \models (\dagger)\\ &\Leftrightarrow N[x] \models \mathcal{C} \Vdash (\dagger).\end{aligned}$$

(The second equivalence is by absoluteness of Σ^1_1 formulas, the fourth one is by homogeneity of $\mathcal{C}$.)

Thus

$$\exists y \subseteq p_0\ \phi(x,y) \Leftrightarrow N[x] \models \mathcal{C} \Vdash \exists y \subseteq p_0\ \phi(x,y).$$

(2) Follows from (1), see [**Je**]. □

11. The second lemma is as follows.

LEMMA. *Suppose that $N \prec H_\lambda$ is countable, $\epsilon > 0$ is rational, $c \in \mathcal{I}^\epsilon_{\mathbb{L}} \cap N$ and $T \in \mathbb{L} \cap N$. Then there exists a Borel set B, coded in N, such that $\mu(B) \leq \epsilon$ and for any $x \in \mathrm{Ra}(N) \setminus B$ there is $(N,\mathbb{L})$-generic $S \leq T$, such that $S \Vdash x \in \mathrm{Ra}(N[G]) \setminus \bigcup_n c_n$.*

PROOF. We drop sub(p)scripts standing by $\mathcal{I}$. Let $\mathcal{D}$ be the family of all open dense subsets of $\mathbb{L}$. Define $C \subseteq {}^\omega 2$ by $x \notin C$ iff

$$\begin{aligned}&\exists S \leq T\ \exists\ f: \mathcal{I} \cap N \to \omega,\ f(c) = 0,\\ &\forall D \in \mathcal{D} \cap N\ \exists R \in D^* \cap N\ S \leq R,\\ &S \Vdash \forall\ d \in \mathcal{I} \cap N\ \forall\ m \geq f(d)\ x \notin d_m.\end{aligned}$$

Easily, if $x \notin C$ then S witnessing this fact is $(N,\mathbb{L})$-generic and $S \Vdash x \in \mathrm{Ra}(N[G]) \setminus \bigcup_n c_n$. (To see genericity find for a given $D \in \mathcal{D} \cap N$, $R \in D^* \cap N$ such that $S \leq R$. By density, D is nonempty below S, so, by Lemma 2, there is $s \in S$ with $R_s \in D \cap N$.)

We are done by the following two claims.

CLAIM 1. $\mu^*(C) \leq \epsilon$.

PROOF. Let $\mathcal{D} \cap N = \{D_n : n \in \omega\}$. Repeat the construction from Lemma 3 in such a way that each $S_n \in N$. Let $S^\dagger = \bigcap_n S_n$. Then for each $D \in \mathcal{D}$ there is $R \in D^* \cap N$ with $S \leq R$ (in particular $S \in D^*$).

For $f : \mathcal{I} \to \omega$ let

$$C_f = \{x \in {}^\omega 2 : S^\dagger \Vdash \exists\ d \in \mathcal{I} \cap N\ \exists\ m \geq f(d)\ \ x \in d_m\}.$$

Let

$$C^\dagger = \{x \in {}^\omega 2 : \forall\ f : \mathcal{I} \to \omega\ f(c) = 0 \Rightarrow x \in C_f\}.$$

Note that $C \subseteq C^\dagger$ (show $x \notin C^\dagger \Rightarrow x \notin C$!).

By Lemma 6, for any $f : \mathcal{I} \to \omega$ we have

$$\begin{aligned}\mu^*(C_f) &\leq \sum_{d \in \mathcal{I} \cap N} \sum_{m \geq f(d)} \mu(d_m)\\ &= \sum_{d \in \mathcal{I} \cap N} \epsilon \cdot 2^{-f(d)}.\end{aligned}$$

It follows that

$$\mu^*(C^\dagger) \leq \inf_f \mu^*(C_f) \leq \epsilon$$

(the only restriction on f is that $f(c) = 0$). $\square$

CLAIM 2. *There is a Borel set B, coded in N, such that $B \cap \mathrm{Ra}(N) = C \cap \mathrm{Ra}(N)$.*

PROOF. Work with the transitive collapse N^* of N in order to apply Lemma 10. To see that

$$S \Vdash \forall\, d \in \mathcal{I} \cap N\ \forall m \geq f(d)\ x \notin d_m$$

is a Δ_0 proposition about x, S and f (with parameters in N^*) note that

$$\begin{aligned} S \not\Vdash x \notin d_m &\Leftrightarrow \exists S' \leq S\ S' \Vdash x \in d_m \\ &\Leftrightarrow \exists S' \leq S\ x \in d_m(S') \\ &\Leftrightarrow \exists s \in S\ x \in d_m(S_s) \\ &\Leftrightarrow \exists R \in \mathrm{dom}(d_m)^* \cap N\ \exists s \in S\ S \leq R\ \&\ x \in d_m(R_s). \end{aligned}$$

(For the third $\Rightarrow$ use $S \in \mathrm{dom}(d_m)^*$ and Lemma 5.) $\square$

12. We are ready for the proof of Theorem 9.

PROOF OF THEOREM 9. Fix a countable $N \prec H_\lambda$, rational $\epsilon > 0$, $c \in \mathcal{I}^\epsilon_{\mathbb{L}} \cap N$, $T \in \mathbb{L} \cap N$, a descending sequence $\langle T_n : n \in \omega\rangle \in (\prod_n \mathrm{dom}(c_n)) \cap N$ below T and $x \in \mathrm{Ra}(N) \setminus \bigcup_n c_n(T_n)$. Apply Lemma 11 uniformly to $\langle c_{n+i} : i \in \omega\rangle$'s and T_n's and get a sequence $\langle B_n : n \in \omega\rangle$ of Borel sets, which is coded in N, such that for each n, $\mu(B_n) \leq \epsilon/2^n$ and if $x \notin B_n$ then there is $S_n \leq T_n$, $(N, \mathbb{L})$-generic, such that $S_n \Vdash x \in \mathrm{Ra}(N[G]) \setminus \bigcup_i c_{n+i}$. Since $\bigcap_n B_n$ is a null Borel set, which is coded in N, $x \in \mathrm{Ra}(N)$ yields $x \notin \bigcap_n B_n$. Fix n with $x \notin B_n$. Then $S_n \Vdash x \in \mathrm{Ra}(N[G]) \setminus \bigcup_{m \geq n} c_m$. Since for $m \leq n$, $c_m(T_m) = c_m(T_n) = c_m(S_n)$, from $x \notin \bigcup_{m \leq n} c_m(T_m)$ we get $S_n \Vdash x \notin \bigcup_{m \leq n} c_m$. Thus $S_n \Vdash x \in \mathrm{Ra}(N[G]) \setminus \bigcup_m c_m$. $\square$

13. Similar arguments work for some other tree forcings, e.g., Miller's rational perfect forcing, Shelah's $\mathbb{Q}_{f,g}$ forcing etc..

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