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Heating and stretching Riemannian manifolds

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Associated to a complete, finite volume hyperbolic surface, there are several classical spectral objects: Selberg's zeta function, the determinant of the Laplacian, the Green's function, 'small' eigenvalues of the Laplacian, and the heat kernel. The behavior of these objects as the hyperbolic metric *degenerates* is the subject of several papers: [**Hjl84**], [**Wlp87**], [**Hjl90**], [**Ji93**], [**Lnd93**], [**Ji95**], [**JrgLnd95**], [**HntJrgLnd95**], [**Ji96**], [**DzkJrg97**]. The above spectral quantities are all related, and their combined behavior can be reduced to the behavior of the heat kernel (see, for example, [**Wlp87**], [**Lnd93**], [**Ji96**], and [**DzkJrg97**]). The basic result is that the heat kernel converges to the heat kernel of the degenerated surface on the complement of the *pinched* geodesic(s) [**Ji95**], [**JrgLnd95**]. (Here 'convergence' will mean uniform convergence on compact sets.) The purpose of this note is to isolate the mechanism underlying heat kernel convergence and to find a much more general geometric context for this mechanism.

THEOREM 1. *Let N be an open submanifold of a smooth manifold M. Let g be a complete (smooth) Riemannian metric on N whose Ricci curvature is bounded from below. Let g_i be a sequence of complete (smooth) Riemannian metrics on M such that $g_i|_N$ converges to g. Then the heat kernel H_{g_i} converges to the heat kernel of H_g on $N \times N \times \mathbb{R}^+$.*

We use the word *stretching* to describe the type of metric convergence in the hypothesis of Theorem 1. Indeed, since (N, g) is (Riemannian) complete, the g-distance from any point in N to the complement, $M \setminus N$, is infinite. The distance functions $dist_{g_i}$ converge to $dist_g$, and hence the manifold is being 'stretched apart' at $M \setminus N$.

EXAMPLE 1 (Hyperbolic degeneration of surfaces). Let M be a surface that admits a hyperbolic metric. Let N be the complement of a simple closed curve $\gamma \subset M$. Let U be a tubular neighborhood of γ. Let g be a complete hyperbolic metric on N such that $(U\backslash\gamma, g)$ is isometric to a disjoint union of two hyperbolic *cusps.* Let g_n be a sequence of complete hyperbolic metrics on M such that $g_n|_N \to g$ and such that γ is a geodesic with respect to each g_n. (See Figure 1). Note that we make no assumption on volume or topological type.

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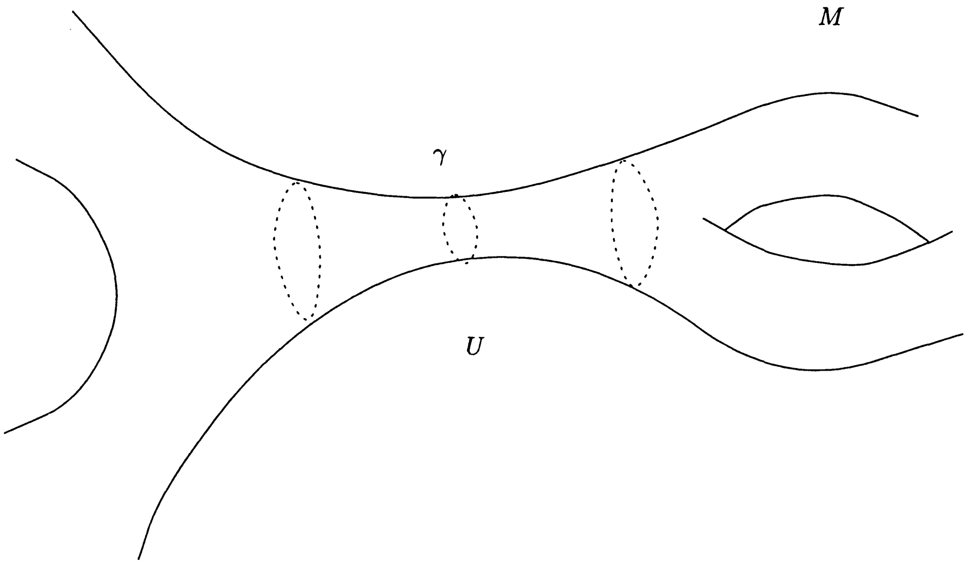

FIGURE 1. Degenerating surface

EXAMPLE 2 (Sphere blown up along circle). Let $M \cong S^2$, and let N be the complement of a simple closed curve. Let g be such that its restriction to each disc induces the complete Euclidean metric. Let g_i be a sequence that converges to g.

EXAMPLE 3 (Unbounded curvature). Let $M \cong N$ be the plane $\mathbb{R}^2$. Let g be the complete Euclidean metric. Let $\kappa : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\kappa|_{[0,1]} \equiv 0$ and $\kappa(r) = -r$ for $r > 2$. Let g_i be the radially symmetric metric whose curvature on the ball of g-radius r is $\kappa(r/i)$. Then g_i converges to g. Note that the curvature of each g_i is not bounded from below.

REMARK 1. In general, the heat kernel is not uniquely defined. In particular, for each metric g_i given in Theorem 1, there might be more than one heat kernel H_{g_i}. On the other hand, the uniqueness of H_g follows from the lower bound on the Ricci curvature of g. (See the discussion in [**Chv**], p. 179, and the references given there.)

The idea behind the proof of Theorem 1 is simple: Although heat diffuses everywhere instantaneously, very little heat is diffused to the far reaches of the manifold in small time. Thus, if M is stretched at $M \backslash N$, then a negligible amount of heat reaches $M \setminus N$ from a given compact subset of N. Hence we may regard a neighborhood of $M \setminus N$ as almost refrigerated. Since the metrics converge uniformly on compact sets, the refrigeration (Dirichlet) problems converge. In sum, the original heat kernels converge.

To make this argument rigorous, one needs 'uniform' decay estimates for the heat kernel H_g of a Riemannian manifold (M, g). We will consider an estimate of the form $|H_g(x, y, t)| \leq f(t, dist_g(x, y))$ holding for all $t > 0$ and $x, y \in M$ where $f : [0, \infty[\times]0, \infty[\to \mathbb{R}$ is continuous. We will abbreviate such an estimate with the notation $H \prec f$.

Before proving Theorem 1, we first prove the more general:

THEOREM 2. *Let N be an open submanifold of a smooth manifold M. Let g be a complete (smooth) Riemannian metric on N whose Ricci curvature is bounded from below. Let g_i be a sequence of complete (smooth) Riemannian metrics on M such that $g_i|_N$ converges to g. Suppose that there exist continuous functions f, f_i, such that $H_g \prec f$, $H_{g_i} \prec f_i$, $f_i \to f$, and $\lim_{d\to\infty} f(t,d) = 0$ for each $t > 0$. Then the heat kernel H_{g_i} (restricted to N) converges to the heat kernel H_g.*

PROOF OF THEOREM 2. Let $\Omega_1 \subset \Omega_2 \subset \cdots \subset N$ be an exhaustion by compact submanifolds with smooth boundary. It will be enough to prove convergence on the sets of the form $\Omega_j \times \Omega_j \times [\delta, T]$ where $0 < \delta < T$.

Let h be a (not necessarily complete) Riemannian metric on N (resp. M), and let H_h^D denote the heat kernel for the Dirichlet problem on (Ω_n, h). For each $x \in \Omega_n$, the function $u_x(y,t) = H_h(x,y,t) - H_h^D(x,y,t)$ is a solution to the heat equation which is C^0 on $\Omega_n \times [0,T]$, C^2 on $\Omega_n \times (0,T]$ and such that $u_x(y,0) \equiv 0$. Hence, by the parabolic maximum principle [**BrsJhnShc**], the maximum of $u_x(y,t)$ over $\Omega_n \times [0,T]$ is achieved on either $\Omega_n \times \{0\}$ or $\partial\Omega_n \times [0,T]$. Since $(H_h)|_{\partial\Omega_n} > 0$ and $(H_h^D)|_{\partial\Omega_n} \equiv 0$, the maximum is achieved on $\partial\Omega_n \times [0,T]$.

Let $\epsilon > 0$. Let d_0 be such that if $d > d_0$, then $f(t,d) < \frac{\epsilon}{6}$. Choose $n > j$ such that $dist_g(\Omega_j, \partial\Omega_n) > d_0$. Since $g_i \to g$ uniformly on Ω_n and $f_i \to f$, there exists $I_1 > 0$ such that for $i > I_1$ and $(x,y) \in \Omega_n \times \Omega_n$, we have $|f_i(t, dist_{g_i}(x,y)) - f(t, dist_g(x,y))| < \frac{\epsilon}{6}$. Since $H_{g_i} \prec f_i$, it follows that for $i > I_1$ and $(x,y,t) \in \Omega_j \times \partial\Omega_n \times [\delta, T]$, we have $|H_{g_i}(x,y,t)| \leq \frac{\epsilon}{3}$.

¿From above we have that the maximum of $|H_g(x,y,t) - H_g^D(x,y,t)|$ (resp. $|H_{g_i}(x,y,t) - H_{g_i}^D(x,y,t)|$) over $\Omega_j \times \Omega_n \times [0,T]$ is achieved on $\Omega_j \times \partial\Omega_n \times [0,T]$. Hence for $i > I_1$ and $(x,y,t) \in \Omega_j \times \Omega_n \times [0,T]$, we have

$$\left|H_g(x,y,t) - H_g^D(x,y,t)\right| < \frac{\epsilon}{3}. \tag{1}$$

$$\left|H_{g_i}(x,y,t) - H_{g_i}^D(x,y,t)\right| < \frac{\epsilon}{3}. \tag{2}$$

To complete the proof, it will be enough to show that the Dirichlet heat kernels converge in C^0. For then there exists $I_2 > I_1$ such that for $i > I_2$ and $(x,y,t) \in \Omega_n \times \Omega_n \times [\delta, T]$, we have $|H_{g_i}^D(x,y,t) - H_g^D(x,y,t)| < \frac{\epsilon}{3}$. This estimate combined with (1) and (2) would give the desired estimate on $\Omega_j \times \Omega_j \times [\delta, T]$.

By way of P. Li's eigenfunction estimates, Chavel and Feldman obtained a C^0 bound on the Dirichlet heat kernel which depends only on the volume and isoperimetric constant of the domain (see [**Chv**] Theorem 10, Chap. IV). Thus, since g_i converges to g, the set of functions $\{H_{g_i}^D\}$ is uniformly bounded in C^0. The convergence g_i to g also gives a uniform bound on the ellipticity constant of Δ_{g_i}. Therefore, one may apply the (parabolic version of) DiGiorgi-Nash theory[1] (See [**Nsh58**] and §14.9 of [**Tyl**]). We find that $\{H_{g_i}^D\}$ is uniformly bounded in a Hölder space C^α for some $\alpha \in]0,1[$. Thus, by compactness, we have a convergent subsequence in C^β for $\beta \in]0,\alpha[$. The limit is a (weak) heat kernel for (N,g), and hence uniqueness—Ricci is bounded from below—implies that the limit equals H_g^D. Moreover, any subsequence of $\{H_{g_i}^D\}$ has a further subsequence that converges to H_g^D in C^β. In particular, the Dirichlet heat kernels $H_{g_i}^D$ converge to H_g^D in the C^0 norm as desired. □

[1]Strictly speaking, the DiGiorgi-Nash estimates are interior estimates, but in the context of the proof, we may always pass to a slightly larger domain.

PROOF OF THEOREM 1. Let (M, h) be an arbitrary Riemannian manifold. Let $\Phi_h(x, r)$ denote the isoperimetric constant of the ball of h-radius r centered at $x \in M$. We have Estimate (2.23) of [**ChgGrmTyl82**] (see also [**Chv**] Chapter VIII.)

$$(3) \qquad |H(x,y,t)| \leq C \cdot (t^{-\frac{n}{2}} + r^{-n})(\Phi_h(x,r)\Phi_h(y,r))^{-\frac{1}{2}} e^{-\frac{s(x,y,r)^2}{4t}}.$$

where $s(x,y,r) = \max\{0, dist_h(x,y) - 2r\}$ and C is a constant that depends only on the dimension of M.

By results from §4 of [**ChgGrmTyl82**] (and Croke's inequality), a lower bound on the Ricci curvature of g implies that there exists $r > 0$ for which $\inf_{x \in N} \Phi_g(x, r) > 0$. Let $K \subset N$ be compact. Since $g_i \to g$ uniformly on K, the sequence $\inf_{x \in K} \Phi_{g_i}(x, r)$ converges to $\inf_{x \in K} \Phi_g(x, r)$. Thus, with h equal to g, g_i, respectively, the right hand side of (3) defines f, f_i, that satisfy the hypotheses Theorem 2. □

REMARK 2. The estimates of [**ChgGrmTyl82**] apply to more general integral kernels than the heat kernel. The work of [**ChgGrmTyl82**] refined and generalized the heat kernel estimates of [**ChnLiYau81**]. Sharper estimates are given in [**LiYau86**] under the strict assumption that Ricci curvature is bounded from below. Moreover, the estimates of [**LiYau86**] involve the Ricci tensor and C^0 convergence does not necessarily imply the convergence of the Ricci tensor.

The proof of Theorem 2 easily generalizes to give:

THEOREM 3. *Let (N, g) be a complete, open, Riemannian manifold with Ricci curvature bounded from below. Let (M_i, g_i) be a sequence of complete Riemannian manifolds. Suppose that there is a sequence of isometric embeddings $\phi_i : N \to M_i$ such that the metrics $\phi_i^*(g)$ converge to g. Then $\phi^*(H_{g_i})$ converges to H_g.*

EXAMPLE 4 (Hyperbolic degeneration of 3-manifolds [**Ji96**] [**DzkJrg97**]). Let (M_i, g_i) be a sequence of closed hyperbolic 3-manifolds in the 'neighborhood' of a complete, finite volume, hyperbolic manifold (N, g) in the sense of Theorem 5.8.2 of [**Thr**]. In particular, M_i is obtained by gluing a solid torus along the boundary of N. This defines ϕ_i and Theorem 3 applies [**Thr**].

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