CONTEMPORARY MATHEMATICS

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Homotopy Methods in Algebraic Topology

Proceedings of an AMS-IMS-SIAM
Joint Summer Research Conference
University of Colorado, Boulder
June 20–24, 1999

J.P.C. Greenlees
with assistance from
Robert R. Bruner
Nicholas Kuhn
Editors

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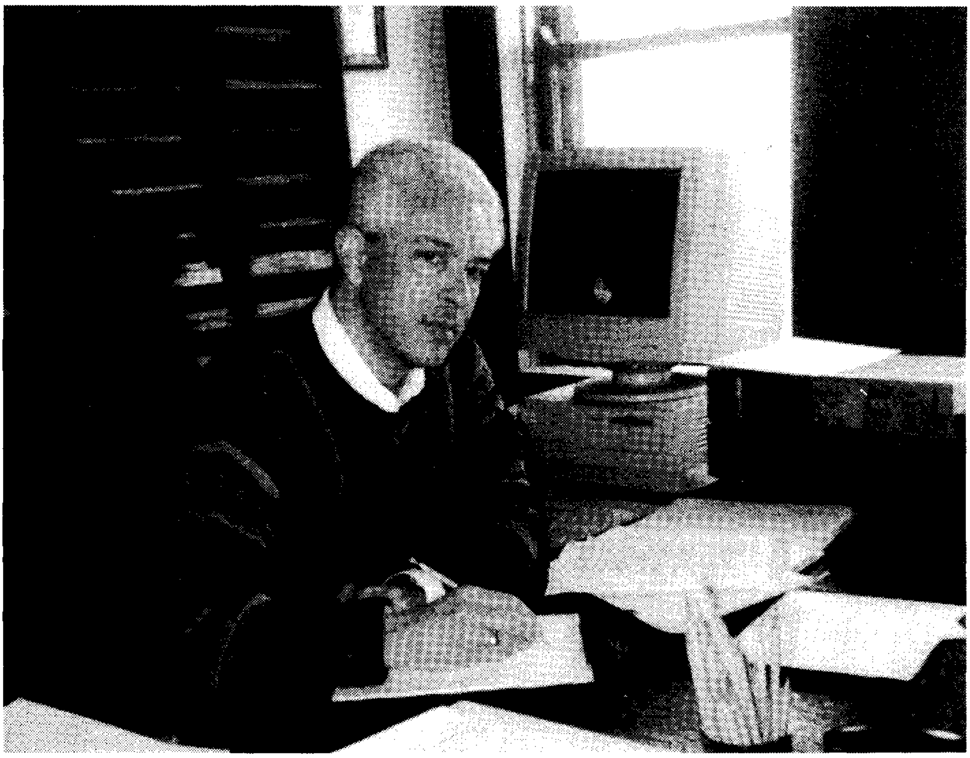
Homotopy Methods in Algebraic Topology



J. Peter May

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J.P.C. Greenlees
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American Mathematical Society
Providence, Rhode Island

This volume contains the proceedings of an AMS-IMS-SIAM Joint Summer Research Conference held at the University of Colorado, Boulder, Colorado on June 20–24, 1999, with support from the National Science Foundation, grant DMS-9618514.

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Preface

This volume is the proceedings of the Conference on Homotopy Methods in Algebraic Topology which took place in Boulder, Colorado from June 20 to June 24, 1999. This was one of the series of AMS–IMS–SIAM Summer Research Conferences held during the summer of 1999 at the University of Colorado. The organizing committee consisted of the three editors of this Proceedings, together with Tony Elmendorf and Jim McClure.

The scientific focus of the conference was on modern aspects of homotopy theory, particularly methods that are being exported to algebraic settings.

This is reflected in the following partial list of topics of the talks: group theory (e.g. by Rickard, Lewis), set theory (Casacuberta), motivic homotopy (Morel, Hu), polynomial functors (Dwyer, McCarthy, Arone, Johnson), elliptic curves (Ando, Hopkins, Mahowald, Rezk), model categories of spectra (Mandell, Karoubi, Shipley), and algebraic K–theory (Madsen, Duflot, Hesselholt). Frank Peterson talked on the history of cobordism theory, a topic connected to the work of many of our speakers.

There were over 100 participants who came from many countries. The 31 speakers reflected this international mix.

The conference coincided with the 60th birthday of University of Chicago Professor J. Peter May. It was not coincidence that four of the organizers were students of Peter, and the fifth a major collaborator. Activities in his honor included a talk *A brief introduction to the work of J. Peter May* by Igor Kriz, and a conference banquet. Among those who attended the conference were 14 of Peter's former Ph.D. students, 5 current students, and numerous other collaborators and ex University of Chicago instructors. This attests to his extraordinary role as an advisor over the last quarter century, and to the unwavering enthusiasm and energy he has brought to the subject. The article by Kriz in this proceedings describes Peter's wide–ranging and influential research, and includes a recent bibliography.

The conference was primarily supported by the National Science Foundation via its funding of the Joint Summer Research Conferences. We also wish to thank the Mathematics Department of the University of Chicago for a generous contribution.

Practical details were smoothly dealt with by the Providence office of the American Mathematical Society and the staff of the University of Colorado. Particular thanks are due to AMS conference coordinator Donna Salter.

Of course, the success of the conference was due to the participants in general and the speakers in particular. This conference proceedings is the result of the efforts of our authors (and referees). To all of you, we offer spirited thanks.

John Greenlees
Bob Bruner
Nick Kuhn

Speakers

- Mathew Ando, University of Illinois
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- Greg Arone, University of Chicago
  Why should loops on Stiefel manifolds stably split?
- Carles Casacuberta, Universitat Autònoma de Barcelona
  Cohomological localizations exist under large cardinal axioms
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  Functors from spaces to spectra
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  Metrics of positive scalar curvature and assembly maps
- Neil Strickland, University of Sheffield
  Gross-Hopkins duality

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- Matt Ando
- Greg Arone
- Anthony Bahri
- Andrew Baker
- Maria Basterra
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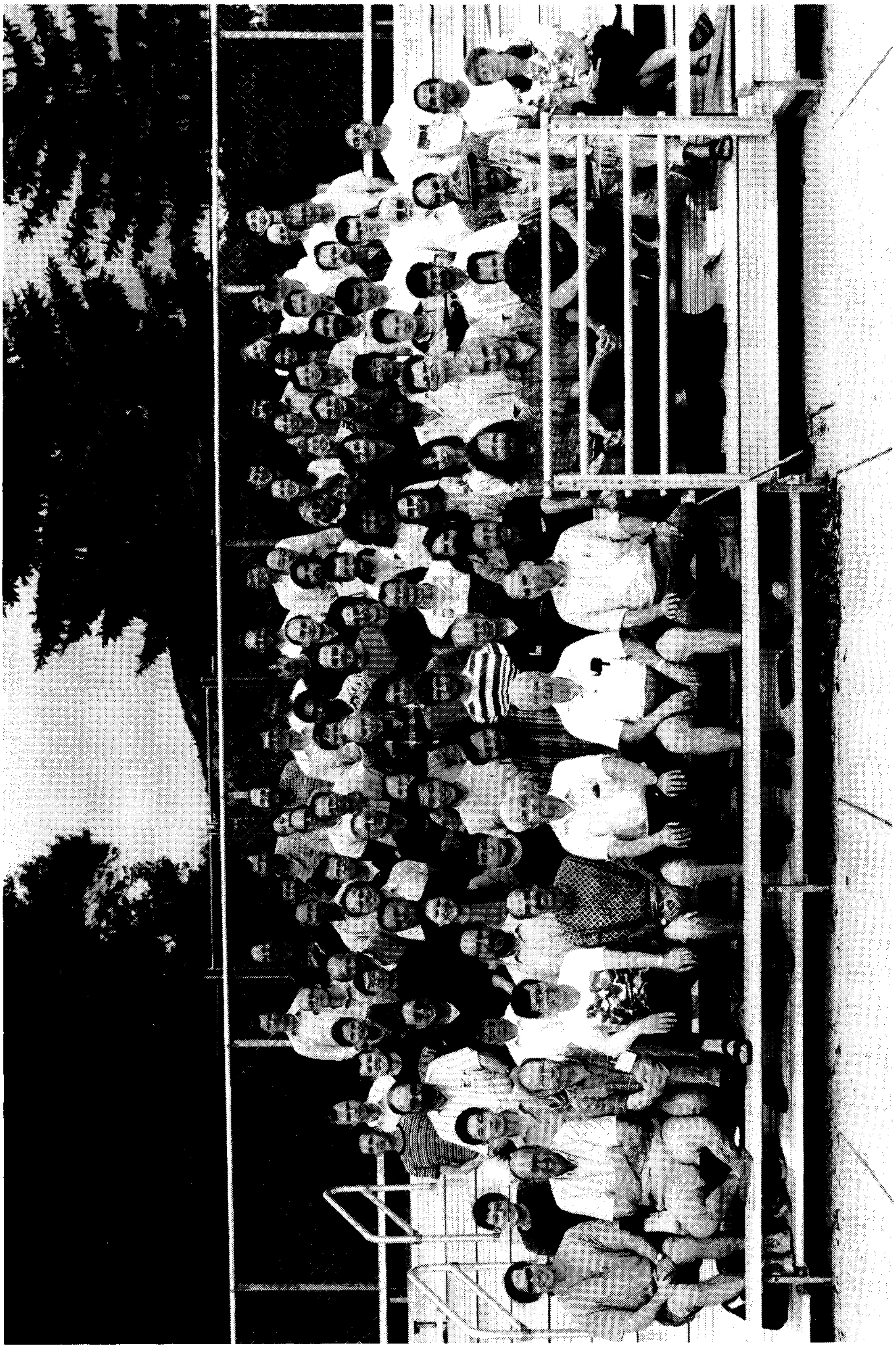

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Row 1: Mike Hopkins, Mark Mahowald, Bob Bruner, Tony Elmendorf, Nick Kuhn, Peter May, Frank Peterson, Fred Cohen

Not pictured: Jeff Caruso, Kathryn Lesh, Al Lundell, Ib Madsen, Haynes Miller, Goro Nishida, Marta Pecuch-Herrero

A BRIEF INTRODUCTION TO THE WORK OF J.PETER MAY, ON THE OCCASION OF HIS 60TH BIRTHDAY

IGOR KRIZ

1. Introduction

This paper is essentially a transcript of a talk I gave at the Boulder conference. Its inadequacy will be obvious: it is almost impossible to describe work as broad as Peter's (work which is still very much in progress) in a lecture or article. With work as central to an area as Peter's, it is also almost impossible, in a reasonable amount of space, to mention all other authors whose work is relevant. Because of that, I deliberately decided against trying to write a historical article: references to other authors are kept to the minimum needed. Let this be an apology to those who are not quoted.

Peter's work helped define the subject of stable homotopy theory. His contributions are both foundational and calculational, and concern virtually all branches of the field. Perhaps his most significant contribution is in the search for concepts. It is my view that very much like natural science, mathematics investigates real phenomena of nature. Unlike the objects of natural science, however, these phenomena are not directly perceptible, but our understanding of them takes the form of concepts. Definitions are used to attain these concepts.

A major portion of Peter's work is precisely in developing the concepts for stable homotopy theory and unifying them if different versions of the concepts exist. Examples are the very notion of a spectrum, infinite loop space theory, and later algebraic structures on spectra. The result of such effort is that in algebraic topology, we have a clear picture of the concepts we are studying. This is necessary in order for an area to be successful: while mathematical intuition may bypass the necessity of a clear picture temporarily, the area ultimately needs a higher degree of resolution to advance.

For the purposes of this paper, I divided Peter's work into 6 areas, partially by subject, partially chronologically. I list the areas below, enumerating their major topics. Peter May's complete bibliography appears in the end. The purpose of the next sections is to provide brief examples of each of the areas, and Peter's contribution to them. These pictures will be by no means complete. However, they should give the reader a flavor of the major directions of Peter's work.

The Steenrod Algebra and related topics Peter's contribution to this area include the May spectral sequence [2], the cohomology of restricted Lie algebras [1], matric Massey products [4], [8], the algebraic approach to Steenrod operations [10], and the cohomology of principal bundles, homogeneous spaces and two-stage Postnikov towers [6], his joint work with Guggenheim [14] and his work on the structure of Hopf algebras [9].

Additive iterated and infinite loop space theory This includes operads, the two-sided bar construction of monads, iterated and infinite loop space theory via operads [7], [12], [18], [28], [29], homology of iterated and infinite loop spaces [11], [17], [39] the work on classifying spaces and fibrations [15], on spectra associated with permutative categories [13], [23], and the joint work of May and Thomason on uniqueness of infinite loop space machines [22], [32].

Multiplicative (infinite) loop space theory The main framework of this theory was worked out by Peter in [20], [41]. A_∞ and H_∞ structures were discussed in [21], [25], [52]. Very interesting topics related to this area include Peter's contributions to the J-homomorphism and chacteristic classes in topological bundles (some of them unpublished).

Equivariant stable homotopy and foundations The fundamental reference on the foundations, including coordinate free spectra, is Peter's joint book with Lewis and Steinberger [53], see also [81], [82]. Other papers of Peter's were on Eilenberg-MacLane G-chomology [36], [72], equivariant localization [37], completion [38], equivariant algebraic K-theory [40], equivariant constructions of non-equivariant spectra [58], and equivariant bundles [45], [55], [65]. A substantial part of Peter's work was related to the Segal conjecture and other completion results in equivariant stable homotopy theory: [42], [43], [50], [51], [54], [59], [60], [61], [62], [63], [71], [73], [80]. The joint paper of Greenlees and May [87] contains a completion theorem for MU-modules, another paper by Greenlees and May studies generalized Tate cohomology [77].

E_∞ **derived categories** The basic reference for commutative and associative smash products is Peter's joint book with Elmendorf, Mandell and myself [83], see also [76], [79]. An algebraic version of this program was published in [78], see also [74], [75], [84], [85], [86]. A major application of the topological program was the Greenlees-May completion theorem for MU, which was published in [87]. More recently, other ways of constructing model categories of spectra with commutative associative smash product emerged ([HSS]), and Peter with his collaborators had a number of results on those, and their unification: [91], [96], [97], [98], [99].

Pedagogical work: So far 30 of Peter's Ph.D. students have graduated and he has 6 current students. He has a number of typed texts about homotopy theory, which he freely distributes to his students. This included notes on homotopical foundations, Hopf algebras, and other topics. Some of the notes were recently published in the form of a text book [95]. Peter also wrote a very nice book about simplicial objects [5].

Many of Peter's significant papers do not specifically fit into any of the above categories, for example the joint work with Zabrodsky on $H^*Spin(n)$, with several collaborators on stable splittings [26], [27], [46], [48], [49], on fibrewise localization and completion [31], on realization of Eilenberg-MacLane spaces as Thom spectra [33], a note on the Bockstein and Adams spectral sequences [34], on the dual Whitehead theorems [47], on characteristic classes in Borel cohomology [56], on a generalization of Smith theory [57] a note on weak equivalences and quasifibrations [67], and many others.

2. The Steenrod Algebra A

The subject of Peter's thesis was the Steenrod algebra. In particular, he introduced a new method of calculating the Ext-groups of the Steenrod algebra, called the *May spectral sequence.* This method revolutionized the subject of calculating stable homotopy groups of spheres: it provided an effective method for calculating the E_2-term of the Adams spectral sequence globally, without working one group at a time.

To fix ideas, let $p = 2$ (Peter's thesis treated the odd prime also). We have the Steenrod algebra A, and its dual

$$A_* = \mathbb{Z}/2[\zeta_i | i \geq 1].$$

This is a Hopf algebra with comultiplication

$$\psi(\zeta_i) = \sum_{j=0}^{i} \zeta_j^{2^{i-j}} \otimes \zeta_{i-j},$$

calculated by Milnor [Miln]. On the Steenrod algebra A, we can consider the filtration by powers of the augmentation ideal (which consists of all elements of positive dimension). The associated graded Hopf algebra E^0A is primitively generated. For primitively generated connected Hopf algebras over a field of characteristic $p > 0$, we have the Poincare-Birkhoff-Witt theorem [MMo], [9], which asserts that

$$E^0(A) = V(L), \tag{2.1}$$

where the right hand side denotes the universal enveloping algebra of the *restricted Lie algebra* L of primitive elements of $E^0(A)$. Recall that a restricted Lie algebra is a Lie algebra with an additional operation $r : L \to L$ called restriction, which corresponds to the Frobenius endomorphism $x \mapsto x^p$ in the universal enveloping algebra. The universal enveloping algebra is the free associative algebra modulo the relations $ab - ba = [a, b]$ and $a^p = r(a)$.

Peter constructed an explicit resolution of $\mathbb{Z}/2$ by free $V(L)$-modules, where L is a restricted Lie algebra. We will not write down this resolution explicitly. It later became known as the Koszul resolution. In fact, a Koszul resolution can be written for the Steenrod algebra A itself (see [Pri]). This resolution is known as the λ-algebra. However, as it turns out, calculations with the λ-algebra are more complicated than with the May resolution.

Peter's method was to calculate the cohomology of $V(L)$, and then use the spectral sequence

$$Ext_{V(L)}(\mathbb{Z}/2, \mathbb{Z}/2) \Rightarrow Ext_A(\mathbb{Z}/2, \mathbb{Z}/2). \tag{2.2}$$

The right hand side is the E_2-term of the Adams spectral sequence for computing stable homotopy group of spheres completed at $p = 2$. (2.2) is the May spectral sequence.

To calculate the left hand side of (2.2), Peter considered the sub-Hopf algebras of A_* given by

$$A_*^n = \mathbb{Z}/2[\zeta_i | i \le n].$$

Then consider the sequence

$$A_*^{n-1} \to A_*^n \to A_*^n // A_*^{n-1} \tag{2.3}$$

where the right hand term denotes the quotient of commutative Hopf algebras. Dualizing (2.3), we obtain a sequence

$$B_{n-1} \leftarrow B_n \leftarrow C_n.$$

Then we have Cartan-Eilenberg spectral sequences

$$(2.4) \qquad {}_nE_2^{p,q} = H^p(E^0B_{n-1}) \otimes H^q(E^0C_n) \Rightarrow H^{p+q}(E^0B_n).$$

This is an inductive method for calculating $H^*(E^0A)$. The spectral sequences (2.4) tend to collapse to early terms. This is the calculational advantage of Peter's approach.

3. Additive iterated and infinite loop space theory

In [12] Peter developed his version of iterated loop space theory, with an approach related to J. Stasheff's theory of homotopy associativity [Sta], R.J.Milgram's models for $\Omega^n\Sigma^nX$ [Milg] and especially J.M. Boardman and R. Vogt's work [BV]. Peter's approach is very geometric, and for a certain point of view gives the best insight. Segal gave a rather different approach to infinite loop spaces [Seg]. The equivalence of all these infinite loop space machines was later proved by May and Thomason in [22].

Peter's version of iterated loop space theory is based on his notion of *operad.* Peter claims to have caused the word *monad* to be used in Saunders MacLane's book [Mac] instead of the older term triple, to mesh with operad.

For rigorous definition of operad, we refer the reader to [12]. Briefly, an operad $\mathcal{C}$ consists of spaces $\mathcal{C}(n)$, $n = 0, 1, 2, \ldots$, with right Σ_n-action (the symmetric group), a unit map $1 \in \mathcal{C}(1)$ and 'composition' structure maps of the form

$$\mathcal{C}(n) \times (\mathcal{C}(k_1) \times \ldots \times \mathcal{C}(k_n)) \to \mathcal{C}(k_1 + \ldots + k_n).$$

A basic example of an operad is the *endomorphism operad* $\mathcal{E}(X)$

$$(3.1) \qquad Hom(X^n, X)$$

of an object X of a symmetric monoidal category. In this case, the Σ_n-action is by permutation of factors, unit is the identity, and composition is defined in the obvious way by composition of maps. One can reconstruct the diagrams required in the definition of operad by examining the commutative diagrams which exist in the specific case (3.1).

An $\mathcal{C}$-algebra for an operad $\mathcal{C}$ is a space X together with a map of operads $\mathcal{C} \to \mathcal{E}(X)$, i.e. in particular an 'action' map

$$\mathcal{C}(n) \times X^n \to X.$$

A *monad* is a functor C from a category Cat into itself, with natural transformations $CC \to C$ and $Id \to C$ which satisfy the obvious associativity and unit properties. A basic example of monad is $C = RL$ where $L : Cat \to Cat'$ is a functor left adjoint to a functor R.

A C-algebra is an object X together with a map $CX \to X$ satisfying associativity and unit diagrams. Any category of universal algebras, i.e. topological spaces with given prescribed operations satisfying prescribed identities, is a category of C-algebras for a canonical monad C in the category of topological spaces. In particular, for an operad $\mathcal{C}$, there is a canonical monad C in the category of based spaces such that $\mathcal{C}$-algebras are the same thing as C-algebras. It is customary to denote this monad by a roman letter which is the same as the script letter denoting the operad.

The notion of C-algebra may be generalized into notions of left and right C-functors $D : Cat' \to Cat$ and $E : Cat \to Cat''$, which have 'action' maps (natural transformations)

$$CD \to D$$

and

$$EC \to E,$$

with associativity and unit diagrams with respect to C.

In this situation, Peter May made extensive use of the *two-sided bar construction*

$$B(E, C, D), \tag{3.2}$$

which is a priori a simplicial object in the category of functors

$$Cat' \to Cat'' \tag{3.3}$$

whose n'th simplicial term is EC^nD [12]. However, most often we work in situations where Cat'' enjoys some type of simplicial realization functor, in which case it is possible to realize (3.3) into a functor of the form (3.2).

Example: The little n-cube operad $\mathcal{C}_n$: $\mathcal{C}_n(m)$ consists of ordered m-tuples of n-cubes with disjoint interiors inside the unit n-cube [BV]. Let also $\mathcal{C}_\infty$ be the infinite-dimensional little cube operad, where $\mathcal{C}_\infty(m)$ is the union of $\mathcal{C}_n(m)$ under the natural inclusion maps.

Now consider the corresponding monad C_n. Let Σ^n and Ω^n denote the adjoint functors in the category of based spaces of n-fold suspension and n-fold looping. Then one observes that for a space X, $\Omega^n X$ is naturally a C_n-algebra, which results in a map of monads

$$C_n(X) \to \Omega^n\Sigma^n(X). \tag{3.4}$$

Theorem 3.5. *(The approximation theorem* [12], [17]*): The map (3.4) is a group completion (equivalence if X is connected). This means that (3.4) induces a group completion on $\pi_0(X)$, and localization away from $\pi_0(X)$ (i.e. inverting $\pi_0(X)$) in homology.*

Theorem 3.6. *(The recognition principle* [12],[18]*): If Y is a C_n-space, then*

$$Y \to \Omega^n B(\Sigma^n, C_n, Y)$$

is a group completion (equivalence if Y is connected).

An important feature is that these theorems can be stabilized. In the stable case, $\mathcal{C}_\infty$ can be replaced by any *E_∞-operad*, which means an operad $\mathcal{C}$ where $\mathcal{C}(m)$ is contractible and Σ_m-free. A space which is an algebra over an E_∞ operad is called an E_∞ space.

The stable version of Theorem 3.6 says that if Y is an E_∞ space, then there is a naturally defined spectrum E and a group completion $Y \to E_0$. The space E_0 is called an *infinite loop space*. In particular,

$$Z \mapsto [Z, E_0],$$

where the right hand side denotes the set of homotopy classes of maps, is the 0-th functor of a generalized cohomology theory on CW-complexes. The group completion is an equivalence if Y is connected, or more generally *group-like*, which means that $\pi_0(Y)$ is a group.

This method of constructing generalized cohomology theories can be demonstrated easily on the case of *permutative categories* [13], [23], which are categories Cat with strictly associative unital operation $\oplus$ and a natural isomorphism

$$\sigma : X \oplus Y \xrightarrow{\cong} Y \oplus X$$

where $\sigma^2 = Id$ and we have commutative diagrams

$$\begin{tikzcd} 0 \oplus X \arrow[rr, "\sigma"] \arrow[dr, "="'] & & X \oplus 0 \arrow[dl, "="] \\ & X & \end{tikzcd}$$

$$\begin{tikzcd} X \oplus Y \oplus Z \arrow[rr, "1\oplus\sigma"] \arrow[dr, "\sigma"'] & & X \oplus Y \oplus Z \arrow[dl, "\sigma\oplus 1"] \\ & Y \oplus X \oplus Z. & \end{tikzcd}$$

Segal [Seg] had noted that such categories give rise to Γ-spaces and thus to spectra. Peter gave a simple proof that would later lead to a multiplicative elaboration to bipermutative categories.

Theorem 3.7. [23] *Let Cat be a permutative category. Then there is an E_∞-operad $\mathcal{D}$ such that the classifying space BCat is naturally a $\mathcal{D}$-algebra. Consequently, there is a canonical group completion of BCat which represents the 0-th functor of a generalized cohomology theory.*

An example of permutative category is given as follows: let R be any ring. Then let the objects of Cat be non-negative integers, and let the morphisms from m to n be isomorphisms $R^m \to R^n$. The cohomology theory given by Theorem 3.7 in this situation is the algebraic K-theory of R.

By using certain standard categorical constructions, the theorem can be extended to a more general and familiar class of *symmetric monoidal categories.*

We can sketch the proof of Theorem 3.7 here. $\mathcal{D}(n)$ is the Čech resolution of Σ_n (i.e. its m-th simplicial term is

$$\underbrace{\Sigma_n \times \ldots \times \Sigma_n}_{m+1}).$$

The $\mathcal{D}$-algebra structure on $BCat$ is given as follows: A typical element of

$$B(Cat^n)_p$$

is a p-tuple

$$(\gamma_1, \ldots, \gamma_p) \tag{3.8}$$

where γ_j is an n-tuple of Cat-arrows

$$\begin{pmatrix} \gamma_{1j} \\ \ldots \\ \gamma_{nj} \end{pmatrix}$$

where $\gamma_{i,j+1}$ and $\gamma_{i,j}$ are composable. A typical element of

$$\mathcal{D}(n)_p$$

is a $p+1$-tuple

$$(\sigma_0, \ldots, \sigma_p) \tag{3.9}$$

of elements of Σ_n. Then the operad action maps (3.8) and (3.9) to the element

$$(\delta_1, \ldots, \delta_p) \in BCat_p$$

where

$$\delta_j = \sigma_{j-1}(\oplus \gamma_j)\sigma_j^{-1}.$$

This gives the requisite $\mathcal{D}$-action because simplicial realization commutes with products.

4. Multiplicative loop space theory

Multiplicative infinite loop space theory studies very strong types of ring structure on generalized cohomology theories. The strongest such structure is captured in the concept of E_∞-*ring spectrum*, due to Peter in collaboration with F. Quinn and N. Ray and this is the subject of the present section. Basically, E_∞ ring spectra are commutative rings in as rigid a sense as possible in the category of spectra. The homology of an E_∞ ring spectrum has of course a graded commutative ring structure, but also a complicated system of cohomology operations investigated by Peter in collaboration with F.R. Cohen and T.J. Lada in [17].

For an E_∞ ring spectrum E, the 0-th space enjoys the structure of an E_∞ *ring space* [20], [41]. Conversely, Peter has a functor assigning to an E_∞ ring space Y an E_∞ ring spectrum E, together with a group completion $Y \to E_0$ [20]. Further, for a symmetric bimonoidal category Cat (i.e. a category with two operations $\oplus$ and $\otimes$ which are commutative, associative, unital and distributive up to appropriately coherent isomorphisms) Peter found a functorial model of $BCat$ which is an E_∞ ring space, although that requires an elaborate correction [41] of the proof in [20].

An interesting feature of an E_∞ ring space Z is that it specifies another 'multiplicative' E_∞ structure (and hence generalized cohomology theory) on the space $Z_\otimes$ which is the union of those components of Z which correspond to multiplicatively invertible elements of $\pi_0(Z)$. For example, for the 0-space QS^0 of the sphere spectrum, $QS^0_\otimes$ is the space F which is the stabilized monoid of self-homotopy equivalences of spheres. The space BF classifies spherical fibrations, and its infinite loop structure comes from the smash product of spherical fibrations. There are also oriented versions of these infinite loop spaces. For connective orthogonal kO-theory, whose 0-space is $BO \times \mathbb{Z}$, $BO_\otimes$ is the 0-space of another generalized cohomology theory called $kO_\otimes$.

We will briefly describe one substantial application of multiplicative infinite loop space theory, namely mod p characteristic classes of topological bundles. We consider the fibration

$$SF \xrightarrow{i} F/Top \xrightarrow{q} BSTop \xrightarrow{Bj} BSF \tag{4.1}$$

Here $STop$ is the union of the groups $STop(n)$ of based oriented self-homeomorphisms of n-spheres under the obvious inclusion maps (suspension). Similarly, SF is the union of the monoids of based oriented self-homotopy equivalences $SF(n)$ of S^n. The problem is calculating the cohomology of $BSTop$, which means calculating characteristic classes of $STop$-bundles.

When localized at $p = 2$, Browder, Liulevicius and Peterson [BLP] showed that

$$MTop_{(2)} \simeq \prod K(\mathbb{Z}/2, n),$$
$$MStop_{(2)} \simeq \prod K(\mathbb{Z}/2, n) \times \prod K(\mathbb{Z}_{(2)}, n).$$

Determining the complete structure, in particular the exact number of factors, is much harder: this was done by Madsen and Milgram [MM], [MM1], see also [BMM].

Now complete at $p > 2$. Here Sullivan's amazing theorem asserted that $BSTop$ is homotopically equivalent to the classifying space of kO-oriented spherical fibrations $B(SF, kO)$. This gives a commutative diagram of the form

$$\begin{tikzcd}
SF \arrow[r, "i"] \arrow[d, "="] & F/Top \arrow[r, "q"] \arrow[d, "\simeq"] & BSTop \arrow[r, "Bj"] \arrow[d, "g\simeq"] & BSF \arrow[d, "="] \\
SF \arrow[r, "e"] & BO_{\otimes} \arrow[r] & B(SF; kO) \arrow[r] & BSF.
\end{tikzcd}$$

With Tornehave, Peter proved the following

Theorem 4.2. *([20]): The above diagram lives in the category of infinite loop spaces.*

This leads to a calculation of $H_*(BSTop, \mathbb{Z}/p)$ in the following way: Let r be a generator of $\mathbb{Z}_p^{\times}$ (the multiplicative group of the p-adics). Define J, $BCoker(J)$ as fibers

$$J \longrightarrow BO \xrightarrow{\psi^r - Id} BO$$

$$BCoker(J) \longrightarrow B(SF; kO) \xrightarrow{c(\psi^r)} BO_{\otimes}.$$

Here $c(\psi^r)$ is the "universal cannibalistic class", which sends a kO-oriented spherical fibration (ξ, μ) to the unit of the K-theory of the base that measures the difference between $\psi^r\mu$ and μ. Define $Coker(J) = \Omega BCoker(J)$.

Theorem 4.3. [20] *There are equivalences of infinite loop spaces*

$$SF \simeq J \times Coker(J),$$
$$B(SF; kO) \simeq BCoker(J) \times BO.$$

This theorem summarizes the work of many people: Boardman, Vogt, May, Adams, Peterson, Priddy, Madsen, Snaith, Torenhave, Friedlander, and others. Using homological operations in multiplicative infinite loop space theory, it can be applied to calculate

$$H_*(B(SF; kO), \mathbb{Z}/p) = H_*(BSTop, \mathbb{Z}/p).$$

5. Foundations of equivariant and non-equivariant stable homotopy theory

Peter's most basic foundational contribution was published in the joint book with Lewis and Steinberger [53]. In this book, the authors describe an approach to equivariant stable homotopy theory with a point set level concept of spectra. The method applies equally to non-equivariant stable homotopy theory, and expresses Peter's approach to the subject. The point set level enables a much better understanding of the smash product of spectra (compare the treatment of commutativity and associativity of the smash product in Adams [Ada] to [53]). Even more importantly, it allowed Peter to reformulate his earlier definition of E_∞ ring spectra in a clear conceptual way that foreshadowed their later description as in some sense strict commutative rings in the category of spectra. In fact, this "strictness" became literal in the later 'brave new worlds' (see [83],[96],[97],[98],[99], and the next section), but that theory is impossible to understand without knowing Peter's classical theory first. We present here a brief summary of the most basic concepts of [53].

A non-equivariant coordinatized spectrum, in Peter's approach, is a sequence of spaces Z_n together with explicit homeomorphisms

$$Z_n \xrightarrow{\cong} \Omega Z_{n+1}.$$

This rigidification allows us to interpret point set theoretical concepts for spectra. This is Peter's first major contribution to foundations of stable homotopy theory, which dates back to 1968 [7]. The second substantial ingredient is the notion of *coordinate-free spectrum.* Here, we pick a *universe*, which is an infinite-dimensional real inner product space $\mathcal{U}$ (we assume that $\mathcal{U}$ is a union of an increasing sequence of finite dimensional vector spaces). Then a coordinate free spectrum consists of spaces Z_V for all finite-dimensional subspaces $V \subset \mathcal{U}$, together with homeomorphisms

$$Z_V \xrightarrow{\cong} \Omega^{W-V} Z_W$$

which satisfy a suitable compatibility axiom. Here $W - V$ is the orthogonal complement of V in W.

One advantage of coordinate free spectra is that one can define G-equivariant spectra for a compact Lie group G precisely the same way. The only adjustment is to the notion of a universe $\mathcal{U}$: One lets $\mathcal{U}$ be a sum of infinitely many copies of certain chosen finite-dimensional real representations of G, including the trivial one-dimensional representation. A special role is played by the *complete universe*, in which case we take *all* finite-dimensional representations of G to form $\mathcal{U}$.

G-spectra indexed by the complete universe immediately give G-equivariant cohomology theories indexed by the real representation ring $RO(G)$: for a finite-dimensional representation V, and a G-CW complex X, simply put

$$E^V(X) = [X, Z_V].$$

Here the right hand side denotes based G-homotopy classes of maps. By stability, this definition can be extended to virtual representations.

One can immediately define a strictly commutative and associative smash product of G-spectra, but it involves change of universe: Let E be a spectrum indexed over a universe $\mathcal{U}$, and E' be a spectrum indexed over a universe $\mathcal{V}$. Then it is almost immediate to define a spectrum $E \wedge E'$ indexed over the universe $\mathcal{U} \oplus \mathcal{V}$. To get back to a chosen universe, one introduces, for an isometry $i : \mathcal{U} \to \mathcal{V}$ of universes, a *change of universe* functor i_* from the category of $\mathcal{U}$-spectra to the category of $\mathcal{V}$-spectra. It is left adjoint to the "pull-back" functor i^*. If E and E' are both indexed over $\mathcal{U}$, we can get an internal smash product of E and E' by forming $i_*(E \wedge E')$ where $i : \mathcal{U} \oplus \mathcal{U} \to \mathcal{U}$ is a chosen isometry. Doing this, we lose point set commutativity and associativity, but relatively easily get these properties up to homotopy, at least if E, E' are CW-spectra.

However, the choice of isometry in the previous paragraph led to Peter's next major idea. Boardman had earlier introduced the use of spaces of linear isometries in stable homotopy theory. Peter and his students found a quite different way of exploiting such spaces with his construction of twisted half-smash products. Given a space X together with a map from X to the space of all isometries from $\mathcal{U}$ to $\mathcal{V}$ and a spectrum E indexed over $\mathcal{U}$, the *twisted half-smash product*

$$X \ltimes E$$

gives a way of taking the "union" of the spectra i_*E over all $i \in X$. Doing this is technical; see [53], and Cole's appendix to [83] for a simplification.

Using the twisted half-smash product, we can now define E_∞ ring spectra: Let $\mathcal{L}(n)$ be the space of all isometries from $\mathcal{U}^n$ to $\mathcal{U}$. This is an E_∞ operad. We say that E is an E_∞ *ring spectrum* if it posesses an action of the operad $\mathcal{L}$: this is given by structure maps

$$\mathcal{L}(n) \ltimes E^{\wedge n} \to E$$

with suitable coherence diagram, analogous to those requiied for a space with an operad action.

The book [53] is an encyclopedia of equivariant stable homotopy theory. The first part of the book includes the *Wirthmüller isomorphism* (generalizations of the formula $F(G/H_+, E) \simeq G/H_+ \wedge E$ for G finite), the *Adams isomorphism* (asserting for E free over the complete universe and G finite that $E^G \simeq (i^*E)/G)$) and *geometric fixed points* defined by

$$\Phi^G E = \lim_{\overrightarrow{V}} \Sigma^{-V^G}(E_V)^G.$$

Other parts of the book treat duality, transfer, classification of equivariant bundles and Thom spectra. Throughout the book, many of the theorems (including the above examples) are stated in greater generality than previously known.

Another of Peter's contributions to equivariant stable homotopy theory is his work on the Segal conjecture. Cast in the language of equivariant stable homotopy theory, this asserts that for a finite group G and finite G-space X, we have

$$\pi_G^*(X)_I^\wedge \cong \pi^*(EG \times_G X) \tag{5.1}$$

where $(\cdot)_I^\wedge$ denotes completion with respect to the augmentation ideal of the Burnside ring of G. The rough history of this problem is as follows. When $G = \mathbb{Z}/2$, the problem is equivalent to a nonequivariant conjectrue about certain Ext groups proposed by Mahowald, and in the late 1970's Lin [Lin] [LDMA] proved (5.1) in this non-eqivariant version. Adams, Gunawardena and Miller then proved (5.1) for $G = (\mathbb{Z}/p)^r$ using an elegant algebraic construction due to W.Singer ([AGM] sets up the algebraic framework, the complete proof was given in a preprint).

Previous to this, May and McClure [42] had reduced the proof of the Segal conjecture for a general finite group to the case of its p-subgroups. Thus the scene was set for G.Carlsson [Car], who in 1982 proved the Segal conjecture by reducing the proof of (5.1) for general p-group G to the case of $G = (\mathbb{Z}/p)^r$.

Carlsson's proof relied in essential ways on constructions needing a well equipped equivariant stable category. Peter and collaborators refined and elaborated Carlsson's ideas in [59], [60] and [63]. In particular, Caruso, May and Priddy [59], gave a new equivariant proof of (5.1) for $G = (\mathbb{Z}/p)^r$ ([59] was titled as a sequel to [AGM]). [59] also gives a substantial simplification of Carlsson's argument by using the following observation about elementary abelian groups: Let G be a finite p-group, and let $\mathcal{A}$ be the category of non-trivial elementary abelian p-subgroup of G (morphisms are inclusions). If $G \neq \{e\}$, then $B\mathcal{A}$ is G-contractible. Indeed, if C is central, we have the natural transformations $C \subseteq CA \subseteq A$, which on the level of classifying spaces gives the requisite G-equivariant contraction of $B\mathcal{A}$. This gives a G-equivalence

$$SX \simeq B\mathcal{A}[X],$$

where SX is the singular set of X (i.e. subspace of points with non-trivial isotropy groups), and $\mathcal{A}[X]$ is the category where objects are pairs (A, x) with $A \in \mathcal{A}$, $x \in X^A$, and morphisms $(A, x) \to (B, y)$ occur if $B \subseteq A$ and $y = x$. This allows an inductive approach to the Segal conjecture [59]. Carlsson [Car] also used an inductive approach, but worked with the category of all proper subgroups instead, which makes the induction more complicated.

In work done concurrently with [42], Peter, with G.Lewis and McClure [43], showed how statement (5.1) for all groups could be used to identify the function spectrum $F(BG_+, BH_+)$ for any finite groups G and H. In particular, the rings of stable self maps $\{BG_+, BG_+\}$ had an explicit group theoretic description as a completed 'two sided' Burnside ring $A(G, G)$. Thus, for example, after Carlsson's work, stable splittings of BG could be related to the simple modules for $A(G, G)$. Initially inspired by such splitting problems, there is by now a huge literature on the modular representation theory of these rings: see, e.g., Nischida's work [Nis] (advertised by Peter in [51]), the work of Peter's students [HK], and [MP].

Peter and his collaborators also obtained generalizations of the Segal conjecture [60], [63].

Another substantial contribution of Peter's was his joint paper with John Greenlees [77] on equivariant Tate cohomology. Let G be a compact Lie group, consider the cofibration:

$$EG_+ \to S^0 \to \widetilde{EG}.$$

If E is a G-spectrum, we have the now famous *Tate diagram*

$$\begin{tikzcd} EG_+ \arrow[r] \arrow[d, "\simeq"'] & E \arrow[r] \arrow[d] & \widetilde{EG} \wedge E \arrow[d] \\ EG_+ \wedge F(EG_+, E) \arrow[r] & F(EG_+, E) \arrow[r] & \widetilde{EG} \wedge F(EG_+, E). \end{tikzcd}$$

Here $c(E) = F(EG_+, E)$ is called the *Borel cohomology* theory associated with E, and $\hat{E} = \widetilde{EG} \wedge F(EG_+, E)$ is called the *Tate cohomology* associated with E.

[77] covers many topics, including spectral sequences, generalizations of the Tate diagram for families, and connections with cyclic cohomology and the root invariant.

Example: Let G act freely on $S(V)$, the unit sphere of a G-representation. Denote by $\alpha_V : S^0 \to S^V$ the non-trivial based map, where S^V is the one point compactification of V. Let E be a G-ring spectrum, and let X be a finite G-CW complex. Then

$$\hat{E}^* X = c(E)^* X[\alpha_V^{-1}] = c(E)_* X[\alpha_V^{-1}].$$

In particular, $\hat{E}_* X$ and $\hat{E}^* X$ are periodic. (Here $*$ means $RO(G)$-grading.)

6. Brave New Worlds

Finally, we arrive at Peter's recent breakthrough in stable foundations, namely symmetric monoidal structures on model categories of spectra. The basic reference is [83] (although accounts of other approaches have appeared more recently, see below). I would like to sketch the history of [83] briefly.

[83] started with a joint idea of Peter and myself on constructing a tensor product of E_∞ modules over an E_∞ ring R in algebra. (The algebraic context will be also described below.) Peter saw that this method could be used in topology, which was a project Tony Elmendorf had been working on. By Spring 1993, Elmendorf, May and myself constructed a derived category of E_∞ R-module spectra, even though it involved transfinite iteration of a bar construction "fattening" of R.

All along, Mike Hopkins had his own idea of constructing the derived category, using a stable homotopy analogue of Quillen's approach to homology: this program played a role in his project of constructing an E_∞ structure on E_n (the joke is Mike's), and rigidifying group actions on E_n suggested by Lubin-Tate theory.

Right before the Boston algebraic topology conference in 1993, people both at Chicago and MIT realized that further point set rigidification of their categories is possible. At the Boston conference, Mike Hopkins gave me a note, in which he proved the formula

$$\mathcal{L}(2) \times_{\mathcal{L}(1)} \mathcal{L}(2) \cong \mathcal{L}(3) \tag{6.1}$$

(recall that $\mathcal{L}(n)$ is the space of all isometries from $\mathcal{U}^n$ to $\mathcal{U}$). This formula caused us to scrap completely our existing version of [83] and to start over. For it forced the following definition:

An $\mathbb{L}$-*spectrum* is a spectrum together with an action $\mathcal{L}(1) \ltimes E \to E$ with the obvious associativity and unitality axioms ($\mathcal{L}(1)$ is a monoid). For $\mathbb{L}$-spectra E_1, E_2, put

$$E_1 \wedge_{\mathcal{L}} E_2 = Coeq((\mathcal{L}(2) \times (\mathcal{L}(1) \times \mathcal{L}(1)) \ltimes (E_1 \wedge E_2) \rightrightarrows \mathcal{L}(2) \ltimes (E_1 \wedge E_2)),$$

the coequalizer being of the two obvious maps. Hopkins' formula (6.1) implies that this product is commutative and associative, although, strangely, not unital.

On the other hand, $S \wedge_{\mathcal{L}} S = S$. Thus, we can define an S-module to be an $\mathbb{L}$-spectrum E for which $S \wedge_{\mathcal{L}} E = E$. Then S-modules with the product $\wedge_{\mathcal{L}}$ form a symmetric monoidal category. We realized that very shortly after Boston. What was not clear at all was how to do homotopy theory in this context. This involved a major contribution of Mike Mandell. Mandell also contributed certain nice applications to [83], for example E_∞ algebraic K-theory.

As mentioned above, the algebraic version of this project (i.e. E_∞-modules over E_∞-algebras in the category of chain complexes) was also realized. The resulting paper [78] also became an accompanying paper of [BK]. If $\mathcal{A}$ is Bloch's higher Chow complex (whose homology groups are the higher Chow groups), then [78] converts $\mathcal{A}$ into an E_∞ algebra. We proposed *mixed Tate motives* as the derived category of $\mathcal{A}$ in the above sense. On the abstract level, much of [83] has algebraic analogues, although, strangely, the theory of unital S-modules was never worked out algebraically (and it is even possible that its algebraic analogue doesn't exist). Our category of mixed Tate motives is 'right' in the sense that it agrees with the derived category of mixed Tate motives obtained from Voevodsky's theory. A part of [78] which doesn't have a topological analogue is rational homotopy theory: in this context, E_∞ structures can be rigidified. For example, from a rational E_∞ algebra, one can manufacture a graded-commutative DGA. We also worked out a t-structure on rational mixed Tate motives, and showed that they form a derived category of an abelian category, assuming the "$K(\pi,1)$-conjecture" of Beilinson, which says that the commutative DGA model of $\mathcal{A}$ is equivalent to the deRham complex of a $K(\pi,1)$ space.

As mentioned above, other symmetric monoidal model categories of spectra were discovered, most notably a category of Jeff Smith [HSS] based on *symmetric spectra* and another based on *orthogonal spectra.* Orthogonal spectra, just like S-modules, can be used also G-equivariantly for a compact Lie group G. This does not have an obvious analogue in symmetric spectra. A *unification result* of all known 'symmetric monoidal spectra machines' was obtained by Mandell, May, Shipley and Schwede [96], [97].

Last but not least, let us mention an application of the results of [83], namely the Greenlees-May completion theorem for MU-modules [87]. Let R be a commutative S-algebra, let $\alpha \in \pi_*(R)$. Let

$$K(\alpha)$$

be the fiber of the localization map $R \to R[\alpha^{-1}]$. Now let $I = (\alpha_1, ..., \alpha_n)$ be a finitely generated ideal in $\pi_*(R)$. Then put

$$K(I) = \bigwedge_{i=1}^{n}{}_R \; K(\alpha_i).$$

It can be shown that $K(I)$ is independent of the choice of generators.

Now let M be an R-module. Define its *local cohomology* by

$$\Gamma_I(M) = K(I) \wedge_R M$$

and its *local homology* by

$$M_I^\wedge = F_R(K(I), M).$$

Now let R be a G-equivariant S-algebra. Suppose I is a finitely generated ideal contained in

$$J_G = Ker(R_*^G \to R_*).$$

Then

$$R[\alpha^{-1}]$$

is contractible non-equivariantly, so $K(I)$ is non-equivariantly equivalent to R. This gives a map

$$K : EG_+ \wedge R \to K(I),$$

which induces maps

$$EG_+ \wedge M \to \Gamma_I(M), \tag{6.2}$$

$$(M_G)^\wedge_I \to F(EG_+, M). \tag{6.3}$$

Theorem 6.4. *(Greenlees, May [87]) Suppose G is a finite extension of a torus, $R = MU$. Then, for any sufficiently large finitely generated ideal contained in J_G, (6.2) and (6.3) are equivalences of MU_G-modules.*

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Mathematical ancestry of J. Peter May

- Karl Jacobi, 1825, Humboldt
- Wilhelm Scheibner, 1848, Halle
- William Story, 1875, Leipzig
- Solomon Lefschetz, 1911, Clark
- Norman Steenrod, 1936, Princeton
- George W. Whitehead, 1941, Chicago
- John Moore, 1952, Brown
- Peter May, 1964, Princeton

Students of J. Peter May

- Neal Glassman, 1968
- Albert F. Lawrence, 1969
- Claude Schochet, 1969
    - Andre Deutz, 1985
- Stanley Kochman, 1970
    - Jesus Mayorquin, 1985
- Ib Madsen, 1970
    - Marcel Bökstedt, 1979
        * Ivar Ottesen, 1997
        * Morten Brun, 1998
    - Erkki Laitinen, 1979
    - Steffen Bentsen, 1983
    - Jan-Alve Svensen, 1986
    - Lisbeth Fajstrup, 1992
    - Kenneth Hansen, 1992
    - Lars Hesselholt, 1994
    - Christian Shlichtkrull, 1997
- Marta Herrero, 1972
- Frederick Cohen, 1972
    - Dan Waggoner, 1985
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- Robert Bruner, 1977
  - Dilip Bayen, 1994
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- Robert Wellington, 1977
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J. P. C. Greenlees, University of Sheffield, UK, Editor, with assistance from Robert R. Bruner, Wayne State University, Detroit, MI, and Nicholas Kuhn, University of Virginia, Charlottesville

This volume presents the proceedings from the AMS-IMS-SIAM Summer Research Conference on Homotopy Methods in Algebraic Topology held at the University of Colorado (Boulder). The conference coincided with the sixtieth birthday of J. Peter May. An article is included reflecting his wide-ranging and influential contributions to the subject area. Other articles in the book discuss the ordinary, elliptic and real-oriented Adams spectral sequences, mapping class groups, configuration spaces, extended powers, operads, the telescope conjecture, p-compact groups, algebraic K theory, stable and unstable splittings, the calculus of functors, the E_∞ tensor product, and equivariant cohomology theories. The book offers a compendious source on modern aspects of homotopy theoretic methods in many algebraic settings.

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