CONTEMPORARY MATHEMATICS

282

Groupoids in Analysis, Geometry, and Physics

AMS-IMS-SIAM Joint Summer Research Conference
on Groupoids in Analysis, Geometry, and Physics
June 20–24, 1999
University of Colorado, Boulder

Arlan Ramsay
Jean Renault
Editors

American Mathematical Society

http://dx.doi.org/10.1090/conm/282

Selected Titles in This Series

282 **Arlan Ramsay and Jean Renault, Editors,** Groupoids in analysis, geometry, and physics, 2001

281 **Vadim Olshevsky, Editor,** Structured matrices in mathematics, computer science, and engineering II, 2001

280 **Vadim Olshevsky, Editor,** Structured matrices in mathematics, computer science, and engineering I, 2001

279 **Alejandro Adem, Gunnar Carlsson, and Ralph Cohen, Editors,** Topology, geometry, and algebra: Interactions and new directions, 2001

278 **Eric Todd Quinto, Leon Ehrenpreis, Adel Faridani, Fulton Gonzalez, and Eric Grinberg, Editors,** Radon transforms and tomography, 2001

277 **Luca Capogna and Loredana Lanzani, Editors,** Harmonic analysis and boundary value problems, 2001

276 **Emma Previato, Editor,** Advances in algebraic geometry motivated by physics, 2001

275 **Alfred G. Noël, Earl Barnes, and Sonya A. F. Stephens, Editors,** Council for African American researchers in the mathematical sciences: Volume III, 2001

274 **Ken-ichi Maruyama and John W. Rutter, Editors,** Groups of homotopy self-equivalences and related topics, 2001

273 **A. V. Kelarev, R. Göbel, K. M. Rangaswamy, P. Schultz, and C. Vinsonhaler, Editors,** Abelian groups, rings and modules, 2001

272 **Eva Bayer-Fluckiger, David Lewis, and Andrew Ranicki, Editors,** Quadratic forms and their applications, 2000

271 **J. P. C. Greenlees, Robert R. Bruner, and Nicholas Kuhn, Editors,** Homotopy methods in algebraic topology, 2001

270 **Jan Denef, Leonard Lipschitz, Thanases Pheidas, and Jan Van Geel, Editors,** Hilbert's tenth problem: Relations with arithmetic and algebraic geometry, 2000

269 **Mikhail Lyubich, John W. Milnor, and Yair N. Minsky, Editors,** Laminations and foliations in dynamics, geometry and topology, 2001

268 **Robert Gulliver, Walter Littman, and Roberto Triggiani, Editors,** Differential geometric methods in the control of partial differential equations, 2000

267 **Nicolás Andruskiewitsch, Walter Ricardo Ferrer Santos, and Hans-Jürgen Schneider, Editors,** New trends in Hopf algebra theory, 2000

266 **Caroline Grant Melles and Ruth I. Michler, Editors,** Singularities in algebraic and analytic geometry, 2000

265 **Dominique Arlettaz and Kathryn Hess, Editors,** Une dégustation topologique: Homotopy theory in the Swiss Alps, 2000

264 **Kai Yuen Chan, Alexander A. Mikhalev, Man-Keung Siu, Jie-Tai Yu, and Efim I. Zelmanov, Editors,** Combinatorial and computational algebra, 2000

263 **Yan Guo, Editor,** Nonlinear wave equations, 2000

262 **Paul Igodt, Herbert Abels, Yves Félix, and Fritz Grunewald, Editors,** Crystallographic groups and their generalizations, 2000

261 **Gregory Budzban, Philip Feinsilver, and Arun Mukherjea, Editors,** Probability on algebraic structures, 2000

260 **Salvador Pérez-Esteva and Carlos Villegas-Blas, Editors,** First summer school in analysis and mathematical physics: Quantization, the Segal-Bargmann transform and semiclassical analysis, 2000

259 **D. V. Huynh, S. K. Jain, and S. R. López-Permouth, Editors,** Algebra and its applications, 2000

258 **Karsten Grove, Ib Henning Madsen, and Erik Kjær Pedersen, Editors,** Geometry and topology: Aarhus, 2000

(*Continued in the back of this publication*)

Groupoids in Analysis, Geometry, and Physics

CONTEMPORARY MATHEMATICS

282

Groupoids in Analysis, Geometry, and Physics

AMS-IMS-SIAM Joint Summer Research Conference
on Groupoids in Analysis, Geometry, and Physics
June 20–24, 1999
University of Colorado, Boulder

Arlan Ramsay
Jean Renault
Editors

American Mathematical Society
Providence, Rhode Island

The 1999 AMS-IMS-SIAM Joint Summer Research Conference in the Mathematical Sciences on "Groupoids in Analysis, Geometry, and Physics" was held at the University of Colorado, Boulder on June 20–24, 1999, with support from the National Science Foundation, Grant DMS-9618514.

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

2000 *Mathematics Subject Classification.* Primary 22A22, 35S05, 46Lxx, 53C12, 53D50, 57R30, 58H05.

Library of Congress Cataloging-in-Publication Data

AMS-IMS-SIAM Joint Summer Research Conference on Groupoids in Analysis, Geometry, and Physics (1999 : University of Colorado)

Groupoids in analysis, geometry, and physics : AMS-IMS-SIAM Joint Summer Research Conference on Groupoids in Analysis, Geometry, and Physics, June 20–24, 1999, University of Colorado, Boulder / Arlan Ramsay, Jean Renault, editors.

p. cm. — (Contemporary mathematics, ISSN 0271-4132 ; 282)

Includes bibliographical references.

ISBN 0-8218-2042-7 (alk. paper)

1. Groupoids—Congresses. I. Ramsay, Arlan. II. Renault, Jean, 1949– III. Title. IV. Contemporary mathematics (American Mathematical Society) ; 282.

QA181.A67 1999
512′.2–dc21

2001046076
CIP

---

10 9 8 7 6 5 4 3 2 1 06 05 04 03 02 01

Contents

Contemporary Mathematics
Volume 282, 2001

Introduction

Arlan Ramsay and Jean Renault

Groupoids were first noticed by Brandt in his study of quadratic forms. The reader can find a reference to his paper in the article by Alan Weinstein in the present volume. Since then they have been used in other areas, and we simply suggest some writers Weinstein mentions in his bibliography: algebraic geometry (Grothendieck), differential geometry (Ehresmann, Mackenzie), algebraic topology (Brown), group representations (Mackey), mathematical physics (Karasev, Weinstein), non-commutative geometry (Connes), and operator algebras (Renault and Connes). Of course there are many other writers.

We hope that the readers of this volume will go beyond it to look at journal articles on groupoids and their uses, as well as other books already in print, and that some of them will join in. We have found groupoids both useful and interesting in their own right.

While the material in these Proceedings is not comprehensive on the topic of groupoids, even in the three areas mentioned, we believe it will still be useful to a researcher or student interested in acquiring insight and information about the uses of these group-like structures. In some cases, there is symmetry of a nature not quite expressible in terms of groups. Other uses of groupoids can involve something of a dynamical nature. Indeed, some of the main examples come from group actions.

It should be noted that in many situations where groupoids have been used, the main emphasis has not been on symmetry or dynamics issues. For example, a foliation is an equivalence relation and has another groupoid associated with it, called the holonomy groupoid. While the implicit symmetry and dynamics are relevant, the groupoid records mostly the structure of the space of leaves and the holonomy. More generally, the use of groupoids is very much related to various notions of orbit equivalence. The point of view that groupoids describe "singular spaces" can be found in work of A. Grothendieck and is prevalent in the non-commutative geometry of A. Connes.

As with groups, measure theoretic, topological and differentiable structures can be added to the algebraic one, according to the context. The purely topological addition can be suitable for the topological context, but many situations in analysis, geometry or physics demand at least the addition of measures, or an awareness of measures that are implicit. Some facts about groupoids in analysis can be explained using only a measure theoretic structure, but that minimum is more often accompanied by a locally compact topology or a manifold structure. Some readers may be surprised that spaces that are only locally Hausdorff are mentioned, but that latitude is needed in the context of foliations and pseudo-groups of transformations.

Adding another structure requires more care for groupoids than it does for groups, mainly because of the fact that only certain pairs of elements can be multiplied. This feature can be expressed in terms of two maps from the groupoid to the space of units (identities), one selecting the right unit and the other the left unit. These maps are usually referred to as *source* and *range* maps, respectively, and denoted by s and r. (Sometimes, they are s and t, for source and target.) The convention is that the source of the left factor must equal the range of the right factor.

It is to be expected that the level sets of these two maps, often called *fibers for the range and source maps,* enter into the proper formulation of how an additional structure relates to the algebraic structure.

For example, a groupoid does not have a single left Haar measure, but instead a family of measures, each of which gives measure zero to the complement of one of the range fibers. A given groupoid element 'translates' the range fiber over its source to the range fiber over its range, by groupoid multiplication. (If $\gamma' \in r^{-1}(s(\gamma))$, then $\gamma\gamma' \in r^{-1}(r(\gamma))$.) Left invariance just means that the measure in the family that is concentrated on the fiber over its source is carried by this translation to the measure in the family that is concentrated on the fiber over its range. Of course, the family of measures must vary in a Borel fashion in the purely measure theoretic setting and continuously in the locally compact setting.

Beyond the difference in formulation, which is expected for groupoids, we have the fact that the existence of a left Haar system is not automatic. Also, a given left Haar system can be multiplied by a well-behaved function of the form $g \circ s$ to obtain another left Haar system, so the uniqueness result for groups must be weakened considerably for groupoids. On the other hand, many processes can be formulated in terms of the class of the measure, even for groups (two measures are in the same class iff they have the same sets of measure 0). Thus the non-uniqueness is not a great hindrance.

For topological groupoids, the range and source maps need to be open as well as continuous. Without this condition, the set of products of elements of two open sets need not be open, for example.

The notion of smooth or Lie groupoid is particularly interesting. For the groupoid structure to be compatible with the manifold structure, the range and source maps need to be submersions, which is analogous to the requirement that they be open in the topological case. In the smooth case, having the range and source maps be submersions guarantees that the set of pairs in $G \times G$ that can be multiplied is a closed submanifold. We refer the reader to articles in these Proceedings but also to the original articles by J. Pradines and to various books (Landsman, MacKenzie, Paterson, and Reinhart) and to papers mentioned there. Let us point out that Lie groupoids and Lie algebroids occur naturally in various contexts of pseudo-differential calculus (e.g. foliated manifolds, manifolds with boundaries, manifolds with corners).

To finish this introduction, we include a few remarks on the content of the articles for the assistance of the reader.

The article by A. Weinstein appeared first in the *Notices of the American Mathematical Society.* It is a good introduction to the way groupoids provide a more comprehensive context for thinking about symmetry than groups do. One of his examples is that of a finite part of a tiling of the plane.

The other articles are more specifically about one of the three topics of the conference, or relations between two or all of them.

According to the introduction of the book by K. Mackenzie mentioned in the article by A. Weinstein, Ehresmann was the first person to use groupoids in differential geometry. We refer to Mackenzie's book for much further information on groupoids in differential geometry.

The article by A. Haefliger is one of the two that are primarily about geometry. That of Haefliger is about foliations. He explains several of the groupoids that can be associated to a foliation and points out results for which groupoids are important. He puts an emphasis on the ones that are étale. The techniques include sheaves of groupoids, cohomology of groupoids, etc.

The article of I. Moerdijk shows how étale groupoids can be used to give convenient generalizations of ordinary topological spaces and manifolds. He explains G-sheaves, cohomology and other constructions for étale groupoids from the viewpoint of derived categories.

The article by G. Della Rocca and M. Takesaki is primarily in the area of analysis. They are specifically interested in understanding equivalence relations and equivalence classes in the context of classification problems. They expand the usual discussion of classification of representations or operator algebras by including the ways that equivalences can occur as well as the fact of equivalence. That is, they want to have a functor defined on the groupoid that produces the equivalence relation on the concretely constructed objects of interest. They make it clear what the advantages of this approach are by working out the case of UHF algebras in detail.

P. Muhly introduces a new perspective for thinking about Fell bundles over groupoids. The approach is to think of them as subcategories of the category of C*-algebras in which the mappings are C*-correspondences. The discussion centers around bundles based on groupoids, as might be expected from the title.

The article by D. Williams is primarily in the area of groupoids themselves, explaining a general tool for obtaining information about groupoids. He defines the Brauer group of a groupoid, and explains how it fits into various perspectives on groupoids. He describes the behavior of the Brauer group relative to equivalence and extensions, and provides cohomological meanings.

A. Paterson has written about index theory for a well-behaved class of groupoid actions. Thus his article stands at the intersection of geometry and analysis, and has groupoids involved in the formulation of the result.

Since G. Mackey introduced the virtual group philosophy, groupoids have been present in group representation theory and in operator algebras. Four contributions to this volume are directly related to operator algebras.

The article by C. Anantharaman and J. Renault is a short survey on the amenability of groupoids, a notion directly prompted by the group case. One needs to distinguish the case of measured groupoids and the case of topological groupoids. The main problem considered is how amenability is reflected on the associated operator algebras. In the measured case, it is known since the work of R. Zimmer that it is related to the hyperfiniteness of the von Neumann algebras. In the topological context, it is related to the nuclearity of the C*-algebras and has interesting applications to the exactness of group C*-algebras.

As stated earlier, groupoids often appear as symmetries of a system or of an algebra. P.-Y. Le Gall gives a concise survey of the Kasparov KK-theory equivariant with respect to a groupoid. This requires a precise definition, in the C*-algebraic setting, of a G-algebra, when G is a groupoid. The corresponding KK-functor, denoted by $KK_G(A, B)$, has now three variables, namely the groupoid G and the G-algebras A, B. Besides its usual properties in A and B, in particular the existence of the Kasparov product, its functoriality in G is useful in computing K- groups and has applications to the Baum-Connes conjecture.

The two other contributions are concerned with Lie groupoids and their Lie algebroids. B. Monthubert shows, in the case of manifolds with corners, how a pseudo-differential calculus is related to the construction of a Lie groupoid G and how to get an index theorem from it. The smoothing operators have a kernel defined on G and the pseudo-differential operators have a symbol defined on the dual $\mathcal{G}^*$ of the Lie algebroid of G. The tangent groupoid, a notion introduced by A. Connes for that purpose, is a key ingredient in the proof of the index theorem. N. Landsman and B. Ramazan show that it also provides a deformation quantization of the Poisson bracket of $\mathcal{G}^*$, in a C*-algebraic context, but with a definition different from that introduced by M. Rieffel.

DEPARTMENT OF MATHEMATICS, 395 UCB, UNIVERSITY OF COLORADO, BOULDER, COLORADO 80309, U.S.A.

E-mail address: ramsay@boulder.colorado.edu

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ D'ORLÉANS, B.P.6759, 45068 ORLEANS CEDEX 2, FRANCE

E-mail address: Jean.Renault@labomath.univ-orleans.fr

Selected Titles in This Series

(Continued from the front of this publication)

Groupoids in Analysis, Geometry, and Physics

Arlan Ramsay and Jean Renault, Editors

Groupoids often occur when there is symmetry of a nature not expressible in terms of groups. Other uses of groupoids can involve something of a dynamical nature. Indeed, some of the main examples come from group actions. It should also be noted that in many situations where groupoids have been used, the main emphasis has not been on symmetry or dynamics issues. For example, a foliation is an equivalence relation and has another groupoid associated with it, called the holonomy groupoid. While the implicit symmetry and dynamics are relevant, the groupoid records mostly the structure of the space of leaves and the holonomy. More generally, the use of groupoids is very much related to various notions of orbit equivalence. The point of view that groupoids describe "singular spaces" can be found in the work of A. Grothendieck and is prevalent in the non-commutative geometry of A. Connes.

This book presents the proceedings from the Joint Summer Research Conference on "Groupoids in Analysis, Geometry, and Physics" held in Boulder, CO. The book begins with an introduction to ways in which groupoids allow a more comprehensive view of symmetry than is seen via groups. Topics range from foliations, pseudo-differential operators, *KK*-theory, amenability, Fell bundles, and index theory to quantization of Poisson manifolds. Readers will find examples of important tools for working with groupoids.

This book is geared to students and researchers. It is intended to improve their understanding of groupoids and to encourage them to look further while learning about the tools used.

ISBN 0-8218-2042-7

9 780821 820421

CONM/282