

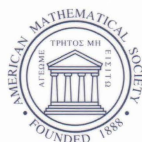
CONTEMPORARY MATHEMATICS

291

q -Series with Applications to Combinatorics, Number Theory, and Physics

A Conference on
 q -Series with Applications to Combinatorics,
Number Theory, and Physics
October 26–28, 2000
University of Illinois

Bruce C. Berndt
Ken Ono
Editors



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American Mathematical Society
Providence, Rhode Island

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This volume contains the proceedings of a conference on q -Series with Applications to Combinatorics, Number Theory, and Physics, which was held at the University of Illinois on October 26–28, 2000.

2000 *Mathematics Subject Classification*. Primary 05Axx, 05Exx, 11Exx, 11Fxx, 11Mxx, 11Pxx, 33Dxx, 33Exx, 81Rxx, 82Bxx.

Library of Congress Cataloging-in-Publication Data

q -series with applications to combinatorics, number theory, and physics : a conference on q -series with applications to combinatorics, number theory, and physics, October 26–28, 2000, University of Illinois / Bruce C. Berndt, Ken Ono, editors.

p. cm. — (Contemporary mathematics, ISSN 0271-4132 ; 291)

Includes bibliographical references.

ISBN 0-8218-2746-4 (alk. paper)

1. q -series—Congresses. I. Berndt, Bruce C., 1939– II. Ono, Ken. III. Contemporary mathematics (American Mathematical Society) ; v. 291.

QA295.Q29 2001
515'.243—dc21

2001053662

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10 9 8 7 6 5 4 3 2 1 06 05 04 03 02 01

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Preface

Those of us who use q -series in our mathematical research are often asked the question, “What is a q -series?” The quickest and simplest (but not so accurate or informative) answer is: “It is a series with q ’s in the summands.” More informatively, we might say q -series contain products $(a; q)_n$, where

$$(0.1) \quad (a; q)_0 := 1, \quad (a; q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad \text{if } n \geq 1.$$

This is not entirely accurate, because in such series one often lets parameters tend to 0 or ∞ , and so products of the type (0.1) may no longer appear. Theta functions frequently arise and so are also thought of as q -series, even though they contain no products of the form (0.1). Lambert series, or generalized Lambert series, often make appearances, especially in applications to number theory, and are also regarded as part of the subject of q -series. In arithmetic applications of modular forms, which include theta functions, one often needs their q -expansions. Thus, a component of the vast theory of modular forms also has a home in the theory of q -series. In conclusion, to paraphrase a senator who once claimed that he could not define pornography, but he knew it when he saw it, most of us working with q -series cannot give a good definition of a q -series, but we know a q -series when we see it.

The subject of q -series can be said to begin with Euler and his pentagonal number theorem. In fact, q -series are sometimes called Eulerian series. Contributions were made by Gauss, Jacobi, and Cauchy, but the first attempt at a systematic development, especially from the point of view of studying series with the products (0.1) in the summands, was made by E. Heine in 1847. In the latter part of the nineteenth and in the early portions of the twentieth centuries, two English mathematicians, L. J. Rogers and F. H. Jackson, made fundamental contributions. Their work was largely ignored by the mathematical community, and so for many years the subject of q -series was considered to be an unimportant, obscure topic on the fringes of respectable mathematics. To illustrate the humble position occupied by the subject for several years, we offer two testimonies. In 1940, G. H. Hardy, on page 222 of his famous book, *Ramanujan*, described what we now call Ramanujan’s famous $1/\psi_1$ summation theorem as “a remarkable formula with many parameters.” This is now one of the fundamental theorems of the subject, but Hardy, as well as other mathematicians during his time, could not foresee its importance. T. W. Chaundy, in his obituary of F. H. Jackson (*J. London Math. Soc.* 37 (1942) 126–128), uncivilly claimed, “The problem [q -series] is very much that of unscrambling an egg.”

Despite such humble beginnings, the subject of q -series has flourished in the past three decades. There are several reasons for this. It took mathematicians several years before most of Rogers and Ramanujan’s contributions to q -series were

appropriately appreciated, and now the subject is a fundamental and active branch of analysis. Many theorems in the theory of hypergeometric series have analogues in basic or q -hypergeometric series, and undoubtedly Chaundy was referring to what he considered to be the esoteric pastime of finding such q -analogues. But irrespective of whether q -analogues exist or not, the subject is replete with beautiful, elegant, and sometimes surprising theorems. Secondly, beginning with the work of Jacobi, q -series have found applications to number theory, and the closeness of these two subjects continues to be ever stronger. Thirdly, increasingly numerous applications to combinatorics are being made, especially in the theory of partitions. Fourthly, through the pioneering work of Rodney Baxter, Barry McCoy, and other theoretical physicists in the past three decades, q -series are now a necessary tool in their subject.

During the year 2000, the Mathematics Department at the University of Illinois embraced *The Millennial Year in Number Theory*. In view of the increasing importance and visibility of q -series, it seemed appropriate that as one of the events in this auspicious *Year*, a conference, *q-series with Applications to Combinatorics, Number Theory, and Physics*, should be held. On October 26–28, sixty-two mathematicians representing twelve countries gathered at the University of Illinois to lecture about and discuss the latest findings. It also seemed appropriate to emphasize survey lectures to help us better chart a course for the future. A total of thirty-nine lectures, highlighted by five plenary survey talks, were given. The plenary lecturers were Scott Ahlgren, George Andrews, Richard Askey, Anne Schilling, and Dennis Stanton. All four aspects (analysis, combinatorics, number theory, and physics) of the subject were featured in these five lectures, as well as in the shorter talks. All the participants helped to make the conference a very successful one, and we thank all of them for their participation. These proceedings contain nineteen of the papers presented at the conference. We hope that they will convey to readers the richness, beauty, and efficacy of the subject.

Special thanks are given to Christian Krattenthaler for a superb evening piano recital. His program can be found before the first paper in this volume.

Many organizations graciously helped to finance the conference. In particular, we are grateful to the National Science Foundation, the National Security Agency, the Number Theory Foundation, The University of Illinois, the Alfred P. Sloan Foundation, and the David and Lucile Packard Foundation for their generous support.

Bruce C. Berndt

Ken Ono

July, 2001

***q*-Series Piano Recital: Levis Faculty Center**

Christian Krattenthaler
8 p.m., Friday, October 27, 2000

PROGRAM

Joseph Haydn (1732–1809)

Sonata in D major, Hob. XVI/37

Allegro con brio
Largo e sostenuto
Finale. Presto ma non troppo

Franz Schubert (1797–1828)

4 Impromptus, D 899

c minor
E flat major
G flat major
A flat major

Intermission

Sergey Rachmaninov (1873–1943) **Morceaux de Fantaisie
(Phantasy Pieces), op. 3**

Élégie
Prélude
Mélodie
Polichinelle
Sérénade

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Despite humble beginnings, the subject of q -series has flourished in the past three decades, particularly with its applications to combinatorics, number theory, and physics. During the year 2000, the University of Illinois embraced *The Millennial Year in Number Theory*. One of the events that year was the conference q -Series with Applications to Combinatorics, Number Theory, and Physics. This event gathered mathematicians from the world over to lecture and discuss their research.

This volume presents nineteen of the papers presented at the conference. The excellent lectures that are included chart pathways into the future and survey the numerous applications of q -series to combinatorics, number theory, and physics.

ISBN 0-8218-2746-4



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